A Modal Characterisation of an Intuitionistic I/O Operation

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Abstract

It is known that Makinson and van der Torre's basic I/O operation out_2 can faithfully be "embedded" into (or "encoded" in) classical modal logic. It is shown that an analogous result holds for the intuitionistic variant of out_2. The target of the embedding is the constructive modal logic CK that evolved through work of Wijesekera, Mendler, de Paiva and Ritter. The very same translation that embeds out_2 into classical modal logic is used.

Keywords: Deontic logic, I/O logic, constructive modal logic CK, logic embedding, intuitionistic logic

1 Introduction

Due to Makinson and van der Torre [14,15], input/output (I/O) logic aims at generalizing the theory of conditional obligation from modal logic [12,13] to the abstract study of conditional codes viewed as relations between classical formulae. The meaning of the normative concepts is given in terms of a set of procedures yielding outputs for inputs. Detachment (or modus ponens) is the core mechanism of the semantics being used. A number of I/O operations are studied in the aforementioned paper [14]. It is shown that they correspond to a series of proof systems of increasing strength. I/O logic belongs to the category of what has been called "norm-based semantics" by Hansen [11, p. 288]. The core idea is to explain the principles of deontic logic, not by some set of possible worlds among which some are ideal or at least better than others, but with reference to a set of explicit norms or existing standards. There are at least two reasons for the recent growth of interest in this approach. First, such a semantics allows one to remain neutral on a number of controversial issues, like the question of whether norms bear truth-values [14], or the question of whether normative statements are based on a maximization process [22]. Second, the norm-based approach has proven to be a fruitful addition to our understanding of key issues in deontic reasoning, like the question of how to

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model permissions [16,5,29], the issue of how to accommodate and resolve conflicts between norms [19], and the question of how to reason about norm violation [15]. As is well-known, these issues highlighted limitations of so-called Standard Deontic Logic (SDL) and its Kripke-type possible worlds semantics, with which philosophers may be more familiar (see, e. g., [24]).

These developments will not be discussed in this paper. For an overview, see [21]. Here I will not go beyond the basic set-up used by Makinson and van der Torre in their [14], except for the following. They use classical propositional logic as the base logic. Parent & al. [20] study the effects of using intuitionistic propositional logic (IPL). It is shown that three of the four standard, classically-based I/O operations have a fully axiomatizable intuitionistic version. These are: the simple-minded I/O operation out_1; the basic I/O operation out_2; and the reusable I/O operation out_3. Of these, the most striking one is undoubtedly out_2. I will be primarily concerned with it. From now onward I will refer to this one as out_2^i, where the superscript i is mnemonic for "intuitionistic". The basic idea is to replace in the semantic idiom the notion of maximal consistent set by its intuitionistic counterpart, the notion of saturated set. The main observation made in [20] is that one obtains the same syntactic characterization of the input/output system, up to the meaning of the connectives. This observation is shown to carry over to the intuitionistic versions of out_1 and out_3. The question of whether it also applies to that of out_4 is left unanswered.

This paper will address another issue left open in [20]. Makinson and van der Torre [14] show that their I/O operations out_2 and out_4 can be reformulated in terms of modal logic. The essential idea is to prefix heads of rules with boxes and apply a suitable modal logic. It is natural to ask if an analog result holds in an intuitionistic setting. The answer to this question turns out to be positive, at least for out_2^i. Admittedly this is a small point, but one (I believe) that is worth clarifying. The intuitionistic modal logic into which out_2^i will be embedded is the system called CK (for constructive K) by Mendler and de Paiva [17] and de Paiva and Ritter [8]. CK is much like (the propositional fragment of) a prior system by Wijesekera [31]. They share the feature that $\Diamond$ does not distribute over disjunction. But CK also rejects the nullary version of the law of distributivity, $\neg\Diamond\bot$, *aka* $\Diamond\bot \rightarrow \bot$. On the semantical side, this is made possible by allowing non-normal (or, as de Paiva and colleagues call them, "fallible" or inconsistent) worlds in the models.

The main result in the paper is a faithful embedding theorem, which echoes the one established by Makinson and van der Torre in the original setting. The theorem is proved for the $\Diamond$-free, first-degree fragment of CK–that is, the subsystem of CK in which formulas do not contain occurrences of $\Diamond$ or nested occurrences of $\Box$. But I will present the full system in order to make the paper self-contained.

Such a result is interesting in its own right, because it makes a bridge between two independent frameworks. This bridge can be used to import results, ideas, and techniques from one to the other. For instance, it can unlock the door to an automation of the source logic. Benzmüller & al. [3] implement the standard I/O operations out_2 and out_4 in Isabelle/HOL [18] via an implementation of their modal translation, making use of the so-called shallow semantical embedding of modal systems K and T into HOL [4]. The embeddings are encoded in Isabelle/HOL for automation.

The layout of this paper is as follows. Section 2 provides the reader with the required background. Section 3 describes the embedding into CK. Section 4 ends with a number of open issues.

2 Background

I start by explaining the basic idea underpinning the I/O framework. In I/O logic, a conditional obligation is represented as a pair (a, x) of propositional formulas, where a is the body (antecedent) and x is the head (consequent). Intuitively, (a, x) may be read as "if a is the case, then x is obligatory". A normative system N is a set of such pairs. Let A be a set of formulas. The main construct has the form: $x \in out(N, A)$. Intuitively this can be read as follows: given input set A (state of affairs), x (obligation) is outputted under norms N.

2.1 Intuitionistic Basic I/O Operation

This section describes the intuitionistic variant of the basic I/O operation out_2 initially put forth by Makinson and van der Torre [14]. The operation is denoted by out_2^i, where the superscript i stands for "intuitionistic". This material is taken from [20].

Throughout this paper, $\mathcal{L}_{\mathsf{IPL}}$ is the set of all formulas in the language of intuitionistic propositional logic. I use the system put forth by Thomason [30]. $\vdash_{\mathsf{IPL}}$ is the derivability relation in this logic. Cn_{IPL} denotes the associated consequence operation, viz. $Cn_{\mathsf{IPL}}(S) = \{a : S \vdash_{\mathsf{IPL}} a\}$, where S is a set of formulas in $\mathcal{L}_{\mathsf{IPL}}$. A set S of formulas is said to be consistent in IPL if there is a wff a such that $S \nvdash_{\mathsf{IPL}} a$.

Definition 2.1 [Saturated set, [30]] Let S be a non-empty set of formulas in $\mathcal{L}_{\mathsf{IPL}}$. S is said to be saturated if the following three conditions hold:

$$S \text{ is consistent in } \mathsf{IPL} \tag{1}$$

$$a \vee b \in S \Rightarrow a \in S \text{ or } b \in S \;(S \text{ is join-prime}) \tag{2}$$

$$S \vdash_{\mathsf{IPL}} a \Rightarrow a \in S \;(S \text{ is closed under } \vdash_{\mathsf{IPL}}) \tag{3}$$

Definition 2.2 implements the notion of (single-step) detachment or modus ponens. It is the *modus operandi* of the semantics.

Definition 2.2 [Image] Let A be a set of formulas. $N(A) = \{x : (a, x) \in N$ for some $a \in A\}$. For $N(A)$, read "the N of A".

Intuitively, $N(A)$ gathers the heads of all the conditional obligations (a, x) in N that are "triggered" by set A. As argued by Boghossian [6], detachment is part of the meaning of a conditional statement. Hence the idea of making detachment the core mechanism of the semantics. [2]

[2] Such a motivation is not in the original papers [14,15]. It is given and discussed in more detail in [22].

Definition 2.3 [out_2^i, intuitionistic basic output]

$$out_2^i(N,A) = \begin{cases} \cap\{Cn_{\mathsf{IPL}}(N(S)) : A \subseteq S, S \text{ saturated}\}, & \text{if } A \text{ is consistent in } \mathsf{IPL} \\ Cn_{\mathsf{IPL}}(h(N)), & \text{otherwise} \end{cases}$$

where $h(N)$ is the set of all heads of elements of N, viz. $h(N) = \{x : (a,x) \in N \text{ for some } a\}$.

Our first observation follows at once from Definition 2.3 and the property of monotony of $\vdash_{\mathsf{IPL}}$. This property tells us that $\Gamma \vdash_{\mathsf{IPL}} x$ whenever $\Delta \vdash_{\mathsf{IPL}} x$ and $\Delta \subseteq \Gamma$.

Fact 2.4 $out_2^i(N,A) \subseteq Cn_{\mathsf{IPL}}(h(N))$.

Put $out_2^i(N) = \{(A,x) : x \in out_2^i(N,A)\}$. This definition leads to an axiomatic characterization that is much like those used for conditional logic. The specific rules of interest here are described below. They are formulated for a singleton input set A (for such an input set, curly brackets will be omitted). The move to an input set A of arbitrary cardinality will be explained in a moment.

$$\text{SI}\ \frac{(a,x) \qquad b \vdash_{\mathsf{IPL}} a}{(b,x)} \qquad\qquad \text{WO}\ \frac{(a,x) \qquad x \vdash_{\mathsf{IPL}} y}{(a,y)}$$

$$\text{AND}\ \frac{(a,x) \qquad (a,y)}{(a, x \wedge y)} \qquad\qquad \text{OR}\ \frac{(a,x) \qquad (b,x)}{(a \vee b, y)}$$

SI and WO abbreviate "strengthening of the input" and "weakening of the output", respectively. IPL is known to be decidable, and thus the relation expressed by each rule is decidable, as is usually required for the rules of an axiom system.

Given a set of rules, a derivation from a set N of pairs (a,x) is a sequence α_1,.., α_n of pairs of formulas such that for each index $0 \leq i \leq n$ one of the following holds:

- α_i is an hypothesis, i.e. $\alpha_i \in N$;
- α_i is $(\top, \top)$, where $\top$ is a zero-place connective standing for 'tautology';
- α_i is obtained from preceding element(s) in the sequence using one of {SI, WO, AND, OR}.

All elements in the sequence are pairs of the form (a,x). Derivation steps done in the base logic IPL are not part of it.

A pair (a,x) of formulas is said to be derivable from N if there is a derivation from N whose final term is (a,x). This will be written as $(a,x) \in deriv_2^i(N)$.

When A is a set of formulas, derivability of (A,x) from N is defined as derivability of (a,x) from N for some conjunction $a = a_1 \wedge ... \wedge a_n$ of elements of A. I understand the conjunction of zero formulas to be a tautology, so that $(\emptyset, a)$ is derivable from N if and only if (iff) $(\top, a)$ is.

Let $deriv_2^i(N,A) = \{x : (A,x) \in deriv_2^i(N)\}$. We have:

Theorem 2.5 (Soundness and completeness) $out_2^i(N,A) = deriv_2^i(N,A)$

Proof. This is [20, Theorem 13]. □

2.2 *Constructive Modal Logic* CK

This section describes the system of constructive modal logic called CK (for constructive K) by Mendler and de Paiva [17] and de Paiva and Ritter [8].

The language is denoted by $\mathcal{L}_{\mathsf{CK}}$. It is obtained by adding to the language of IPL the two modal operators $\Box$ and $\Diamond$. For simplicity's sake, $\bot$ is identified with a privileged atomic sentence, as in so-called minimal logic.

Definition 2.6 A *Kripke model* of CK is a structure $M = (W, \leq, R, v)$, where W is a non-empty set of possible worlds (or points), $\leq$ is a reflexive and transitive binary relation on W, R is a binary relation on W, and v is a function assigning to each propositional letter p a subset of W, viz $v(p) \subseteq W$. Furthermore, $\leq$ is required to be hereditary with respect to propositional variables:

$$\text{If } w \leq w' \text{ and } w \in v(p), \text{ then } w' \in v(p)$$

$\leq$ is used to express the forcing condition for the arrow connective $\to$, whilst R (with a little help from $\leq$) is employed to articulate the forcing condition for the modal operators $\Box$ and $\Diamond$.

Definition 2.7 [Forcing] Given a model $M = (W, \leq, R, v)$, and a world $w \in W$, the forcing relation $M, w \vDash a$ (read as "formula a is 'forced' at world w in model M") is defined by induction on the structure of a using the following clauses:

- $M, w \vDash p$ iff $w \in v(p)$
- $M, w \vDash \top$
- $M, w \vDash b \wedge c$ iff $M, w \vDash b$ and $M, w \vDash c$
- $M, w \vDash b \vee c$ iff $M, w \vDash b$ or $M, w \vDash c$
- $M, w \vDash b \to c$ iff $(\forall w')\ (w \leq w' \Rightarrow (M, w' \vDash b \Rightarrow M, w' \vDash c))$
- $M, w \vDash \Box b$ iff $(\forall w')\ (w \leq w' \Rightarrow \forall u(w'Ru \Rightarrow M, u \vDash b))$
- $M, w \vDash \Diamond b$ iff $(\forall w')\ (w \leq w' \Rightarrow \exists u(w'Ru\ \&\ M, u \vDash b))$

As usual I will drop reference to M, and write $w \vDash a$, when it is clear what model is intended.

A world w is said to be normal if $w \nvDash \bot$, and non-normal (or fallible) if $w \vDash \bot$. The following two constraints are placed on models:

$$\text{If } w \text{ is non-normal and } w \leq w' \text{ or } wRw', \text{ then } w' \text{ is non-normal} \qquad (c_1)$$

$$\text{If } w \text{ is non-normal, then } M, w \vDash p \text{ for all propositional letters } p \qquad (c_2)$$

(c_1) and (c_2) imply that, for all formula a, $M, w \vDash a$, whenever w is non-normal.

Following Fitting [10], Mendler and de Paiva [17] introduce a "hybrid" notion of consequence, which distinguishes between global and local assumptions. Global (or universal) assumptions are required to hold at all points in a given model, while local assumptions are required to hold at a given point in that model. I will use a local consequence relation instead. A formula a is said to be a semantic consequence of A (notation: $A \models a$), whenever, for every model M, and for all worlds w in M, if all of A hold at w, then so does a. My reason for doing so is twofold. First, it will simplify the arguments. Second, the contrast between global and local assumptions will not play any role in subsequent developments.

CK comes with a Hilbert-style proof system, whose axioms consist of all the validities of the intuitionistic propositional logic IPL together with

$$\Box(a \to b) \to (\Box a \to \Box b) \tag{K-$\Box$}$$

$$\Box(a \to b) \to (\Diamond a \to \Diamond b) \tag{K-$\Diamond$}$$

CK also has the rule of modus ponens and the rule of necessitation for $\Box$. As usual, $\vdash_{\mathsf{CK}} a$ indicates that a is a theorem in CK, and $A \vdash_{\mathsf{CK}} a$ indicates that the formula a is in CK a deductive consequence of the set of (local) assumptions A. We have $A \vdash_{\mathsf{CK}} a$ whenever there is a finite $A' \subseteq A$ such that $\vdash_{\mathsf{CK}} \bigwedge A' \to a$. The limiting case where $\bigwedge \emptyset = \top$ is included.

The soundness and completeness theorem is stated below.

Theorem 2.8 *$A \models a$ iff $A \vdash_{\mathsf{CK}} a$.*

Proof. This is Mendler and de Paiva [17, Theorem 1]. □

3 Modal Embedding Result

The intuitionistic analog of Lindenbaum's lemma will be needed. It reads:

Lemma 3.1 *Let $A \cup \{a\} \subseteq \mathcal{L}_{\mathsf{IPL}}$. If $A \nvdash_{\mathsf{IPL}} a$, then there is a saturated set S of formulas (in $\mathcal{L}_{\mathsf{IPL}}$) such that $A \subseteq S$ and $a \notin S$.*

Proof. This is [30, Lemma 1]. □

The following observation will also come in handy.

Theorem 3.2 *Let A be a non-empty set of formulas in $\mathcal{L}_{\mathsf{IPL}}$. A is consistent in CK if and only if A is consistent in IPL.*

Proof. For the left-to-right direction, suppose A is consistent in CK. By Theorem 2.8, A is satisfiable in a model $M = (W, \leq, R, v)$ of CK. That is, there is a normal world w in M such that $w \models x$ for all $x \in A$. Let $M^w = (W^w, \leq^w, v^w)$, where

- $W^w = \{u \in W : u \text{ is normal } \& \ w \leq u\}$
- $\leq^w = \leq \cap (W^w \times W^w)$
- $v^w(p) = v(p) \cap W^w$ for all propositional letters p

M^w is an ordinary Kripke model of IPL. An easy induction establishes that each world in M^w forces the same formulas $a \in \mathcal{L}_{\mathsf{IPL}}$ as in M. Hence, A is satisfiable in an ordinary Kripke model of IPL. By soundness, A is consistent in IPL.

The proof of the right-to-left direction is similar. Starting with a model M of IPL in which A is satisfiable, one needs to get a model M' of CK in which A is also satisfiable. M' shares W, $\leq$ and v with M. Its new component R is the identity relation. In M', constraints (c_1) and (c_2) are trivially verified, because all the worlds are normal. □

The very same translation that embeds the original I/O logic into classical modal logic is used. The core idea is to convert each pair in N into an intuitionistic implication whose head is prefixed with $\Box$, and then use CK to calculate the output. The main result in this paper is that such an embedding is faithful. The exact statement of the result to be established is given by equation (4) where $N^{\Box} = \{a \to \Box x : (a, x) \in N\}$:

$$x \in deriv_2^i(N, A) \Leftrightarrow h(N) \vdash_{\mathsf{IPL}} x \text{ and } N^{\Box} \cup A \vdash_{\mathsf{CK}} \Box x \tag{4}$$

The left-to-right (LTR) implication says that the translation "preserves" derivability of outputs, while the right-to-left (RTL) implication says that no new outputs can be derived. Below each direction is established in turn.

Theorem 3.3 (Faithfulness, LTR) *If $x \in deriv_2^i(N, A)$, then $h(N) \vdash_{\mathsf{IPL}} x$ and $N^{\Box} \cup A \vdash_{\mathsf{CK}} \Box x$.*

Proof. Assume $x \in deriv_2^i(N, A)$. The claim $h(N) \vdash_{\mathsf{IPL}} x$ follows from Theorem 2.5 and Fact 2.4.

By definition of $deriv_2^i$, $(a, x) \in deriv_2^i(N)$, for a conjunction $a = a_1 \wedge ... \wedge a_n$ of elements in A. One shows that $N^{\Box} \cup \{a\} \vdash_{\mathsf{CK}} \Box x$ by a straightforward induction on the length of the derivation of (a, x):

Base case: (a, x) has a derivation of length 1. In that case, either (a, x) is $(\top, \top)$ or $(a, x) \in N$. The claim $N^{\Box} \cup \{a\} \vdash_{\mathsf{CK}} \Box x$ holds, because each of $\top \to \Box\top$ and $((a \to \Box x) \wedge a) \to \Box x$ is a theorem in CK;

Inductive step: (a, x) has a derivation of length $n+1$. The interesting case is when (a, x) is obtained from earlier lines by a derivation rule. Only two $\Box$-principles are needed. One is the axiom K-$\Box$. It is needed to handle WO. The other is $(\Box a \wedge \Box b) \to \Box(a \wedge b)$. It is needed to handle AND, and is derivable in CK.

The claim $N^{\Box} \cup A \vdash_{\mathsf{CK}} \Box x$ follows from $N^{\Box} \cup \{a\} \vdash_{\mathsf{CK}} \Box x$ and the principle of cumulative transitivity for $\vdash_{\mathsf{CK}}$. This principle tells us that $\Gamma \cup \Delta \vdash_{\mathsf{CK}} y$ whenever $\Gamma \vdash_{\mathsf{CK}} b$ and $\Delta \cup \{b\} \vdash_{\mathsf{CK}} y$. $\Box$

Theorem 3.4 (Faithfulness, RTL) *If both $h(N) \vdash_{\mathsf{IPL}} x$ and $N^{\Box} \cup A \vdash_{\mathsf{CK}} \Box x$, then $x \in deriv_2^i(N, A)$.*

Proof. I show the contrapositive. Assume $x \notin deriv_2^i(N, A)$ and $h(N) \vdash_{\mathsf{IPL}} x$. To show: $N^{\Box} \cup A \nvdash_{\mathsf{CK}} \Box x$. Our aim is to establish that $N^{\Box} \cup A \not\models \Box x$. The desired conclusion, $N^{\Box} \cup A \nvdash_{\mathsf{CK}} \Box x$, follows at once from this and the soundness half of the completeness theorem for CK.

By Theorem 2.5, $x \notin out_2^i(N, A)$. Then $out_2^i(N, A) \neq Cn_{\mathsf{IPL}}(h(N))$, so by Definition 2.3, A is consistent in IPL and $out_2^i(N, A) = \cap\{Cn_{\mathsf{IPL}}(N(S)) : A \subseteq S, S \text{ saturated}\}$. So, since $x \notin out_2^i(N, A)$, there is some saturated set $S \supseteq A$ with $x \notin Cn_{\mathsf{IPL}}(N(S))$. Define $M = (W, \leq, R, v)$ as follows:

- $W = \{w : w \text{ is a saturated set of formulas in } \mathcal{L}_{\mathsf{IPL}}\}$
- $w \leq u$ iff $w \subseteq u$
- wRu iff: for all $(b, y) \in N$, if $b \in w$, then $y \in u$
- $v(p) = \{w : p \in w\}$

M is a model of CK. By construction, $S \in W$. The following observation will come in handy.

Claim 3.5 *Let x be a formula in $\mathcal{L}_{\mathsf{IPL}}$. For all $w \in W$, $x \in w$ iff $M, w \models x$.*

Proof. [Proof of Claim 3.5] By induction on x. I consider only the case where x is a conditional, $b \to c$, focusing on the proof of the right-to-left direction. Assume $b \to c \notin w$. Since a saturated set is closed under $\vdash_{\mathsf{IPL}}$, Definition 2.1, $w \nvdash_{\mathsf{IPL}} b \to c$. By the deduction theorem, $w \cup \{b\} \nvdash_{\mathsf{IPL}} c$. By Lemma 3.1, there is a saturated set u such that $w \cup \{b\} \subseteq u$ and $c \notin u$. On the one hand, $w \leq u$. On the other hand, the inductive hypothesis yields $u \models b$ and $u \not\models c$, which suffices for $w \not\models b \to c$. □

Claim 3.6 below will help us establish the desired intermediate conclusion, viz. $N^{\Box} \cup A \not\models \Box x$.

Claim 3.6 *The following holds in M:*

$$\text{For all } a \in A, S \models a \tag{5}$$
$$\text{For all } b \to \Box y \in N^{\Box}, S \models b \to \Box y \tag{6}$$
$$S \not\models \Box x \tag{7}$$

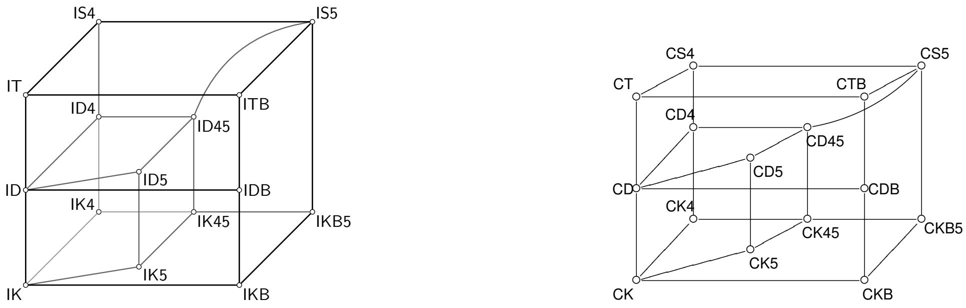
Proof. [Proof of Claim 3.6] (5) follows easily from Claim 3.5 and $A \subseteq S$. For (6), let $b \to \Box y \in N^{\Box}$. Let t be such that $S \leq t$ and $t \models b$. Let u and v be such that $t \leq u$ and uRv. The formula b is in $\mathcal{L}_{\mathsf{IPL}}$. By Claim 3.5, $b \in t \subseteq u$. Since $(b, y) \in N$ and uRv, $y \in v$. By Claim 3.5 again, $v \models y$, since y is in $\mathcal{L}_{\mathsf{IPL}}$ too. By the forcing condition for $\Box$, $t \models \Box y$. By the forcing condition for $\to$, $S \models b \to \Box y$. Hence, for all $b \to \Box y \in N^{\Box}$, $S \models b \to \Box y$.

For (7), recall that $N(S) \nvdash_{\mathsf{IPL}} x$. By Lemma 3.1, there is a saturated set t such that $N(S) \subseteq t$ and $x \notin t$. On the one hand, $t \in W$. On the other hand, x is a formula in $\mathcal{L}_{\mathsf{IPL}}$. So $t \not\models x$, by Claim 3.5. Let $(b, y) \in N$. Suppose $b \in S$. By construction, $y \in N(S) \subseteq t$. Hence, $y \in t$, which suffices for SRt. Trivially $S \leq S$. By the forcing condition for $\Box$, $S \not\models \Box x$ as required. □

This concludes the proof of Theorem 3.4. □

It is worthwhile to mention that the proofs of Theorems 3.3 and 3.4 also go through in Wijesekera's initial system. Thus, the proposed embedding works in both systems. However, the proof of Theorem 3.4 does not carry over to the constructive modal logic CS4 (see, e.g., [1,8]). CS4 is obtained by supplementing CK with the T-axioms $\Box x \to x$, $x \to \Diamond x$ as well as the S4-axioms $\Box x \to \Box\Box x$, $\Diamond x \to \Diamond\Diamond x$. It is characterized by the class of models in which R is in addition reflexive and transitive, and R and $\leq$ are such that $(R \circ \leq) \subseteq (\leq \circ R)$ where $\circ$ denotes composition of relations. In the model M used in the proof of Theorem 3.4, the latter constraint is satisfied. But there is no guarantee that R is reflexive and transitive. Thus there is no guarantee that M is a model of CS4.

One would like to know whether the embedding result extends to other systems in the so-called intuitionistic "modal cube" introduced in [28] or its constructive variant (See Figure 1). For a given system, call it S, to act as a substitute for CK, its $\Diamond$-free, first-degree fragment must coincide with that of CK. I conjecture that this



(a) The intuitionistic cube (cf. [28])

(b) The constructivist cube (cf. [2]).

Fig. 1. The modal cubes.

requirement is at least met for the systems between IK and IK45 in the intuitionistic modal cube, and for the systems between CK and CK45 in its constructive variant. The detailed verification of this claim must be postponed until another occasion.

4 Conclusion

I conclude this paper by highlighting a number of issues to consider in future research besides the aforementioned one.

First, one would like to know if the embedding can be extended to the other intuitionistic I/O operations defined in [20]. The basic reusable I/O operation out_4^i is worth a mention. It is much like out_2^i, except that it also allows outputs to be recycled as inputs. On the syntactical side, we have in addition the rule of cumulative transitivity:

$$\text{CT}\ \frac{(a, x) \qquad (a \wedge x, y)}{(a, y)}$$

Makinson and van der Torre [14] show that the classically based out_4 can faithfully be embedded into a number of modal systems containing the T-axiom. It would be pleasant to be able to report that an analogous result holds for out_4^i, if one uses, e.g., the propositional fragment of Fitch's $\mathcal{M}$ [9] which is CK plus the T-axioms $\Box x \to x$, $x \to \Diamond x$. However, the fact that out_4^i still lacks an axiomatic characterization analogous to Theorem 2.5 presents a serious obstacle to obtaining such a result.

Second, I have confined myself to unconstrained I/O logic, which is usually considered just a stepping stone towards a finer-grained account of normative reasoning. The present account inevitably inherits the problems faced by unconstrained I/O logic, which have led to the further developments alluded to in the introductory section. In particular the present account puts aside the subtleties of contrary-to-duty (CTD) obligations. This can be illustrated with the "white fence" scenario due to Prakken and Sergot [27]: there should no fence; if there is a fence, it should be white; there is a fence. The encoding in CK gives: $N^\Box$ is $\{\top \to \Box\neg f, f \to \Box(w \wedge f)\}$ and A is $\{f\}$. One derives $\Box\bot$, which is the opposite of what we want. Drawing on analogous constructions in the logics of belief change and nonmonotonic inferences, the traditional approach in I/O logic consists in constraining the I/O operations to avoid output that is inconsistent with the input [15]. However, the systems of constrained I/O logic do not have a known axiomatic characterization. Furthermore, the (full join and meet) constrained I/O operations are in general nonmonotonic with respect to the input set A. It is unclear how they can be encoded in CK, whose consequence relation is monotonic. An alternative approach to CTDs has recently been studied in Parent and van der Torre [22,23,26,25]. The unconstrained I/O operations are defined in such a way that they are not closed under the consequence relation of the base logic. Furthermore, some of these I/O operations have a built-in consistency check, which filters out excess output. This yields variant proof systems with neither the rule WO nor the zero-premise rule TAUT: [3]

$$\text{TAUT}\ \frac{-}{(\top, \top)}$$

The question remains open whether these variant systems have an intuitionistic counterpart that can be embedded into some existing (non-normal) constructive modal logic(s) or variant thereof [7].

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