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Maximality vs. Optimality in Dyadic Deontic Logic

Completeness Results for Systems in Hansson's Tradition

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Abstract This paper reports completeness results for dyadic deontic logics in the tradition of Hansson's systems. There are two ways to understand the core notion of best antecedent-worlds, which underpins such systems. One is in terms of maximality, and the other in terms of optimality. Depending on the choice being made, one gets different evaluation rules for the deontic modalities, but also different versions of the so-called limit assumption. Four of them are disentangled, and compared. The main observation of this paper is that, even in the partial order case, the contrast between maximality and optimality is not as significant as one could expect, because the logic remains the same whatever notion of best is used. This is established by showing that, given analogous properties for the betterness relation, the same system is sound and complete with respect to its intended modelling. The chief result of this paper concerns Åqvist's system **F** supplemented with the principle (CM) of cautious monotony. It is established that, under the maximality rule, **F**+(CM) is sound and complete with respect to the class of models in which the betterness relation is required be reflexive and smooth (for maximality). From this, a number of spin-off results are obtained. First and foremost, it is shown that a similar determination result holds for optimality; that is, under the optimality rule, **F**+(CM) is also sound and complete with respect to the class of models in which the betterness relation is reflexive and smooth (for optimality). Other spin-off results concern classes of models in which further constraints are placed on the betterness relation, like totalness and transitivity.

Keywords Conditional obligation · Preference-based semantics · Strong completeness · Optimality · Maximality · Limit assumption

1 Introduction

The present paper is concerned with so-called preference-based semantics for dyadic deontic logic. These rely on ranking possible worlds in terms of a binary relation of comparative goodness or betterness. Structures of this sort seem to have made their first explicit appearance in print with the paper of Hansson [13]. There they are used to give a semantic

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analysis of contrary-to-duty (or secondary) obligations, which tell us what comes into force when some other (primary) obligations are violated.

Noticeably, Hansson only uses the flat (i.e. non-nested) fragment of the language, and the systems he proposes (he confidently calls them "dyadic standard systems of deontic logic" - DSDL) are purely semantical. Truth-functional compounds of deontic and non-deontic sentences (mixed formulas, for short) are prohibited, and so are iterations of deontic modalities. The semantics is, then, given in terms of valuations of Boolean formulas, and a preference relation over valuations. Three systems, DSDL1, DSDL2, and DSDL3, are considered. They are obtained by requiring the betterness relation to fulfil conditions of increasing strength. In DSDL1, the relation is only reflexive. In DSDL2, it also satisfies the so-called limit assumption. In DSDL3, the relation is also transitive and total (or strongly connected).

Results in the literature so far have mostly concerned the meta-theory of DSDL3. First, Spohn [32] has found a weakly complete and decidable axiomatization for the flat fragment of DSDL3 over the valuations semantics discussed by Hansson. Åqvist [2, 3] relaxes the two restrictions placed by Hansson on the language of DSDL3. He also supplements the latter one with alethic modalities. He, then, provides a complete axiomatization **G** of the resulting language over a Kripke-type possible worlds semantics. He develops a method of his own, called the "systematic frame constants" method, which has roots in Lewis [20]. Hansen [9, p. 130] shows that the method fails with respect to strong completeness. In Parent [25], the strong completeness of **G** is established using an alternative method. In Hansen [12], it is also shown that DSDL3 (as axiomatized by Spohn) is (only) weakly complete with respect to so-called (uniquely prioritized) imperative semantics.[1]

The overall theme of this paper is to establish strong completeness results for dyadic deontic logics in the tradition of Hansson's systems, especially DSDL2 and DSDL3. It is an often overlooked fact that two different accounts of conditional obligation can be plugged into such systems. They may be called the "optimality" and the "maximality" rules, respectively. Following Hansson, within the preference-based semantics, the truth conditions for the deontic modalities are typically phrased in terms of best antecedent-worlds. The statement "it ought to be the case that B given A" (in symbolic notation, $\bigcirc(B/A)$) is true if the best A-worlds are all B-worlds. However, the notion of best A-worlds can in turn be characterized in terms of either "optimization" or "maximization". The distinction between the two is well-established in rational choice theory [5, 14, 30, 33]. Recently it has been revived by Arló Costa [4] in the context of the study of belief change theory. Little attention has been paid to it in the deontic logic literature. Roughly speaking, while optimizing involves choosing an alternative that is judged to be better than any other, maximizing only requires choosing an alternative that is not judged to be worse than any other. It is striking that, in their reconstruction of DSDL3, Spohn and Åqvist both use optimization, where Hansson uses maximization. Still, the first two continue to use the word "maximality". Nothing in the paper by Spohn suggests that he is aware of the difference. By contrast, Åqvist seems to be quite aware of it. In [2, p. 180], he parenthetically remarks that he disregards the fact that Hansson's own concept of maximality differs from his own. Given Spohn and Åqvist's prime focus on DSDL3, this is all right. For the two notions coincide when the betterness relation is total. Still, the choice between the two matters as long as there are incomparabilities between possible worlds − a situation that cannot be ruled out. This raises the question of

[1] This semantics is in the "imperativist" tradition of deontic logic, which attempts to reconstruct deontic logic as a logic about imperatives. For a list of authors working in this tradition, see, e.g., [10, fn. 1].

how each account compares to the other, and what their respective commitments are, when "gaps" in the ranking are allowed.

In this connection, this paper will contrast models in which deontic formulas are understood in terms of maximality, and models in which they are interpreted in terms of optimality. The aim is to provide an axiomatization for each class of models, so they can be compared in terms of postulates. Establishing the completeness of the identified systems is the chief purpose of this paper. I shall primarily, but not exclusively, be interested in classes of models where the betterness relation is a partial order. The fact that the aforementioned limit assumption comes in several forms will add an additional complication to the picture. A secondary aim of this paper is to disentangle these forms, and compare them. Another secondary theme is to clarify the role of the assumption of totalness.

The plan of this paper is as follows. Section 2 sets the stage by introducing the framework being used. The main result is stated and proved in Section 3. Some auxiliary results are presented in Section 4. Section 5 sums up the main findings.

2 Logical Framework

2.1 Language

In addition to a set $\mathbb{P}$ of propositional variables and the usual Boolean sentential connectives, the language has the following characteristic primitive logical connectives : the alethic modal operators $\Box$ (for necessity) and $\Diamond$ (for possibility) ; and the two dyadic deontic operators $\bigcirc(-/-)$ and $P(-/-)$, which may be read as *"It ought to be that ..., given that ..."* and *"It is permitted that ..., given that ..."*, respectively. The set $\mathscr{L}$ of well-formed formulas (wffs) is defined in the usual way. There are no restrictions as to iterations of dyadic deontic operators and modal ones. As usual, $\bigcirc A$ and PA are short for $\bigcirc(A/\top)$ and $P(A/\top)$.

2.2 Semantics

I start by defining the notions of optimality and maximality more precisely,[2] since their contrast will have a bearing on the semantics itself.

Let $\succeq$ denote a non-strict (or weak) binary relation ranging over a set X of items x, y, ... X can be viewed as the universe of discourse. Intuitively, $\succeq$ ranks members of X in terms of comparative goodness or betterness. "$x \succeq y$" can be read as "x is at least as good as y". $\succeq$ is said to be reflexive over X iff $x \succeq x$ for all $x \in X$. It is said to be transitive over X iff, for all $x,y,z \in X$, whenever $x \succeq y$ and $y \succeq z$, then $x \succeq z$. $\succeq$ is said to be total (complete, or strongly connected) over X iff, for all $x,y \in X$, either $x \succeq y$ or $y \succeq x$. Items x and y are said to be incomparable, written $x||y$, whenever $x \not\succeq y$ and $y \not\succeq x$. $\succ$ denotes the strict relation induced by $\succeq$, defined by $x \succ y$ iff $x \succeq y$ and $y \not\succeq x$. "$x \succ y$" can be read as "x is strictly better than y". The reader may easily verify that reflexivity of $\succeq$ implies irreflexivity of $\succ$ (i.e., for all $x \in X$, $x \not\succ x$), and that transitivity of the $\succeq$-type implies transitivity of the $\succ$-type.

Let $\emptyset \neq S \subseteq X$. An optimal element of S is a member of S that is at least as good as any other element of this set. Formally:

$$x \in \text{opt}_{\succeq}(S) \quad \Leftrightarrow \quad x \in S \ \& \ x \succeq y \text{ for all } y \in S$$

[2] I follow Sen's terminology in his [30]. The notions of optimization and maximization are respectively referred to as "stringent" and "liberal" maximization by Herzberger [14], and "greatestness" and "maximality" rationality by Suzumara [33].

An optimal element of S is, thus, an upper bound of S that is contained in S.

The notion of maximal element is best defined in terms of a strict order. For a given x in S to qualify as a maximal element of S, no other y in S must be strictly better than x. Formally:

$$x \in \max_{\succ}(S) \Leftrightarrow x \in S \ \& \ \nexists\, y \in S\,(y \succ x)$$

This can be rewritten in terms of $\succeq$ as follows:

$$x \in \max_{\succeq}(S) \quad \Leftrightarrow \quad x \in S \ \& \ \forall y \in S\,(\,y \succeq x \Rightarrow x \succeq y\,)$$

The definition in terms of $\succeq$ makes it clear that $\mathrm{opt}_{\succeq}(S) \subseteq \max_{\succeq}(S)$, although the converse may fail in general. In fact, the basic contrast between optimization and maximization arises from the possibility of "gaps" in the ranking. Put $X = S = \{x,y\}$ with $x \succeq x$ and $y \succeq y$. We have $x||y$, i.e. x and y are incomparable, and so $\mathrm{opt}_{\succeq}(S) = \emptyset$ whilst $\max_{\succeq}(S) = \{x,y\}$. The counterexample may be blocked by requiring one of $x \succeq y$ and $y \succeq x$ to hold. This suggests that totalness is a sufficient condition for the two notions to coincide.

Proposition 1 *If $\succeq$ is total, then* $\max_{\succeq}(S) \subseteq \mathrm{opt}_{\succeq}(S)$.

Proof Assume $\succeq$ is total. Let $x \in \max_{\succeq}(S)$, but $x \notin \mathrm{opt}_{\succeq}(S)$. From the latter, $x \not\succeq y$ for some $y \in S$. From the former, $y \not\succeq x$, which contradicts the opening assumption of totalness. □

The semantic apparatus can now be introduced. A model is a structure $M = (W, \succeq, V)$ in which:

(i) $W \neq \emptyset$ (W, the "universe" of the model, is a non-empty set of possible worlds);
(ii) $\succeq \subseteq W \times W$ is a comparative goodness or betterness relation;
(iii) $V : \mathbb{P} \to \mathscr{P}(W)$ (V is a function assigning to each $p \in \mathbb{P}$ a subset $V(p)$ of W).

I write $M,x \models A$ to indicate that A is true at world x in M. $\models$ is as usual for Boolean connectives. It is defined as follows for the alethic modalities, where $\|A\|^M$ is $\{x \in W : M,x \models A\}$, the "truth-set" of A in M:

$$M,x \models \Box A \quad \text{iff} \quad \|A\|^M = W$$
$$M,x \models \Diamond A \quad \text{iff} \quad \|A\|^M \neq \emptyset$$

The truth conditions for the deontic modalities have the following pattern, where $\mathrm{best}_{\succeq}(\|A\|^M)$ is a shorthand for the set of best (according to $\succeq$) worlds in which A is true:

$$M,x \models \bigcirc(B/A) \quad \text{iff} \quad \mathrm{best}_{\succeq}(\|A\|^M) \subseteq \|B\|^M$$
$$M,x \models P(B/A) \quad \text{iff} \quad \mathrm{best}_{\succeq}(\|A\|^M) \cap \|B\|^M \neq \emptyset$$

However, one gets two different pairs of evaluation rules depending on which of the following two equations is adopted:[3]

$$\mathrm{best}_{\succeq}(\|A\|^M) = \max_{\succeq}(\|A\|^M) \qquad \text{(Max rule)}$$
$$\mathrm{best}_{\succeq}(\|A\|^M) = \mathrm{opt}_{\succeq}(\|A\|^M) \qquad \text{(Opt rule)}$$

[3] Both definitions can be found in the literature. Hansson [13], Makinson [22, §7.1], Prakken and Sergot [28] and Schlechta [29] use the max rule. Alchourrón [1, p. 76], Åqvist [2, 3], Hansen [11, §6], McNamara [23] and Spohn [32] work with the opt rule. Neither Goldman [8], nor Jackson [17], nor Hilpinen [16, §8.5] specifies what notion of best is meant. (The last one uses "best" and "deontically optimal" interchangeably, but leaves optimality undefined.)

I shall say that a model M applies the max rule or the opt rule, depending on whether, in M, deontic formulas are interpreted using the former or the latter. From Proposition 1, it immediately follows that, in a given model M with $\succeq$ total, the same deontic formulas are true at a given world whatever rule is applied.

I shall often drop reference to M when it is clear what model is intended. The properties usually envisaged for $\succeq$ are reflexivity, transitivity, totalness, and the so-called limit assumption. This one will be discussed in the next subsection.

The notion of semantic consequence over some class of models is defined as usual, in terms of truth preservation at a given world in a given model in that class. The notion of validity of a wff A in some class of models is as usual too, and so is the notion of satisfiability of a set of wffs in some class of models. Where Γ is a set of wffs, the notation $\Gamma \models A$ indicates that A is a semantic consequence of Γ with respect to a given class of models, and $\models A$ indicates that A is valid with respect to a given class of models.

2.3 Limit assumption

The exact formulation of the limit assumption varies amongst authors. It can be given two basic forms:

Limitedness

If $\|A\| \neq \emptyset$ then $\text{best}_{\succeq}(\|A\|) \neq \emptyset$ (LIM)

Smoothness (or stopperedness)

If $x \models A,$ then: either $x \in \text{best}_{\succeq}(\|A\|)$ or $\exists y$ s.t. $y \succ x$ & $y \in \text{best}_{\succeq}(\|A\|)$ (SM)

The name "limitedness" is from Åqvist [2, 3], "smoothness" from Kraus & al. [18], and "stopperedness" from Makinson [21]. Each of (LIM) and (SM) may be specified further by identifying $\text{best}_{\succeq}(X)$ with either $\max_{\succeq}(X)$ or $\text{opt}_{\succeq}(X)$. A betterness relation $\succeq$ will be called "opt-limited" or "max-limited" depending on whether (LIM) holds with respect to $\text{opt}_{\succeq}$ or $\max_{\succeq}$. Similarly, it will be called "opt-smooth" or "max-smooth" depending on whether (SM) holds with respect to $\text{opt}_{\succeq}$ or $\max_{\succeq}$.[4]

This gives us four versions of the limit assumption. With the strong assumptions of transitivity and totalness, these different forms of the limit assumption coincide. However, with weaker constraints on $\succeq$, they may well diverge.

Proposition 2

(a) (i) opt-limitedness implies max-limitedness;
(ii) given totalness of $\succeq$, max-limitedness implies opt-limitedness;
(b) (i) opt-smoothness implies max-smoothness;
(ii) given totalness of $\succeq$, max-smoothness implies opt-smoothness.

Proof (a.i) and (b.i) follow from the definitions involved. (a.ii) and (b.ii) follow from Proposition 1. □

[4] Hansson [13] and Prakken and Sergot [28] use max-limitedness, while Alchourrón [1, p. 84], Åqvist [2, 3], Hansen [11, §6], McNamara [23] and Spohn [32] use opt-limitedness, and Schlechta [29] max-smoothness. I am not aware of any authors who have considered opt-smoothness explicitly.

Proposition 3

(a) (i) max-smoothness implies max-limitedness;

(ii) given transitivity and totalness of $\succeq$, max-limitedness implies max-smoothness;

(b) (i) opt-smoothness implies opt-limitedness;

(ii) given transitivity of $\succeq$, opt-limitedness implies opt-smoothness.

Proof Throughout the proof the letter X is used to denote some arbitrarily chosen truth-set $\|A\|$.

(a.i) follows from the definitions involved. For (a.ii), assume $\succeq$ is max-limited, and let $x \in X$ be such that $x \notin \max_{\succeq}(X)$. So there is some $y \in X$ such that $y \succ x$, i.e. $y \succeq x$ and $x \not\succeq y$. By max-limitedness, there is some $z \in X$ with $z \in \max_{\succeq}(X)$. By totalness of $\succeq$, either $z \succeq y$ or $y \succeq z$. Either way, $z \succeq y$, because $z \in \max_{\succeq}(X)$. By transitivity of $\succeq$, $z \succeq x$. But $z \succeq y$ and $x \not\succeq y$. So, by transitivity of $\succeq$ again, $x \not\succeq z$, and thus $z \succ x$, which suffices for max-smoothness.

(b.i) follows from the definitions involved. For (b.ii), assume $\succeq$ is opt-limited, and let $x \in X$ be such that $x \notin \mathrm{opt}_{\succeq}(X)$. So there is some $y \in X$ such that $x \not\succeq y$. By opt-limitedness, there is also some $z \in X$ such that $z \in \mathrm{opt}_{\succeq}(X)$. So $z \succeq x$ and $z \succeq y$. A similar argument as before yields $x \not\succeq z$ using transitivity. Hence $z \succ x$, which suffices for opt-smoothness. $\square$

For the reader's convenience, the relationships just established are represented in an Implication Diagram with the direction of the arrow representing that of implication. The implication relations shown in the picture on the left-hand side hold without restriction. By contrast, those shown on the right-hand side hold under the hypothesis that $\succeq$ meets the property (or pair of properties) displayed as label.

$$\begin{tikzcd} \text{opt-limited} \arrow[d] & \text{opt-smooth} \arrow[l] \arrow[d] \\ \text{max-limited} & \text{max-smooth} \arrow[l] \end{tikzcd}$$

(a) Implications holding unrestrictedly.

$$\begin{tikzcd} \text{opt-limited} \arrow[r, "\text{transitivity}"] & \text{opt-smooth} \\ \text{max-limited} \arrow[u, "\text{totalness}"] \arrow[r, "\text{totalness}", "\text{transitivity}"'] & \text{max-smooth} \arrow[u, "\text{totalness}"'] \end{tikzcd}$$

(b) Converse implications.

Figure 1: Forms of the limit assumption, and their relationships.

Usually the limit assumption is phrased using the same notion of best as in the evaluation rules for the deontic modalities. This has become a common practice, and myself I will follow it. However, there is no obstacle that can stop us from combining different notions of best, as one thinks fit.

One often takes the point of the limit assumption to rule out infinite sequences of strictly better worlds, hence the name.[5] It is natural to ask if, amongst the four conditions just discussed, one implements this requirement better than the others. An in-depth discussion of this issue falls outside the scope of this paper. I shall just point out that, if the strong conditions of transitivity and totalness are relaxed, then there is trouble either way. However, I do not know of any better way to implement the requirement in question, and so we will have to live with that.

[5] However, not all the authors have used it that way. For instance, for Hansson it seems to have been more a concern for non-emptiness, which is essential to validate the principle (given as $D^\star$, §2.4 below) that ought implies permitted, for consistent (or possible) antecedents.

On the one hand, max-limitedness may seem too liberal in itself. For, in the absence of totalness, it may fail to have the desired effect upon infinitely ascending chains. To see this, put $M = (W, \succeq, V)$ with $W = \{x_i : i < \omega\} \cup \{x\}$, $x \succeq x$, $x_i \succeq x_i$ for all $i < \omega$, and $x_j \succeq x_i$ iff $i \leq j$. Note that $x || x_i$ for all $i < \omega$. Assume all the worlds in W are "duplicate" of one another, in the sense that they all satisfy exactly the same wffs. Then, $\succeq$ is max-limited, because $\max_{\succeq}(\|A\|) = \{x\}$, for all wff A such that $\|A\| \neq \emptyset$. Still, W contains an infinite sequence of strictly better worlds.

On the other hand, each of opt-limitedness, opt-smoothness and max-smoothness may seem too stringent. They do more than just rule out infinitely ascending chains, but also exclude models that otherwise seem quite reasonable. For opt-limitedness and opt-smoothness, just go back to the example given immediately before Proposition 1; that is, consider $M = (W, \succeq, V)$ with $W = \{x, y\}$, $x \succeq x$, $y \succeq y$ and $V(p) = W$. Since $x || y$, opt-limitedness is not met for p. Nor is opt-smoothness fulfilled for p, Proposition 3 (b.i). Thus, even though there is no infinitely ascending chain at all in W, such a model is ruled out by either condition. Something quite similar happens with max-smoothness. In the absence of transitivity, it excludes the following variant model: $M = (W, \succeq, V)$ with $W = \{x, y, z\}$, $x \succ y \succ z$ (reflexivity of the $\succeq$-type is left implicit) and $V(p) = W$. Note that $x || z$. Transitivity is not satisfied, because $x \succeq y$ and $y \succeq z$, whilst $x \not\succeq z$. We have $\max_{\succeq}(\|p\|) = \{x\}$. Max-smoothness is not satisfied for p (witness: z).

From the above, it should be apparent that in general finiteness of the universe of the model is not a guarantee of the limit assumption being met. Again, the strong assumptions of transitivity and totalness are needed.

Proposition 4 *Consider a model $M = (W, \succeq, V)$ with W finite, and $\succeq$ reflexive and transitive. In such a model, $\succeq$ is max-smooth, thereby max-limited. If in addition $\succeq$ is total, then $\succeq$ is opt-smooth, thereby opt-limited.*

Proof The argument for max-smoothness is more concisely stated using the "strict" counterparts of the two properties that are required from $\succeq$, i.e. irreflexivity and transitivity of the $\succ$-type. Assume, to reach a contradiction, that max-smoothness is not met. Then, there are A and x_0 such that i) $x_0 \notin \max_{\succeq}(\|A\|)$ and ii) $y \notin \max_{\succeq}(\|A\|)$ for any y with $y \succ x_0$. Using i), it follows that there is some x_1 with $x_1 \models A$ and $x_1 \succ x_0$. By irreflexivity of $\succ$, x_1 is other than x_0. Using ii), it follows that $x_1 \notin \max_{\succeq}(\|A\|)$. So there is some x_2 with $x_2 \models A$ and $x_2 \succ x_1$. Again, by irreflexivity of $\succ$, x_2 is other than x_1. By transitivity of $\succ$, $x_2 \succ x_0$. By irreflexivity of $\succ$, x_2 is other than x_0. Also, the assumption ii) applies again, yielding $x_2 \notin \max_{\succeq}(\|A\|)$. One gets an infinite sequence of worlds (all distinct) by reiterating this argument indefinitely.

The fact that $\succeq$ is max-limited follows from Proposition 3 (a.i). The fact that, given totalness of $\succeq$, $\succeq$ is opt-smooth and opt-limited follows from Proposition 2 (b.ii) and 3 (b.i). $\square$

2.4 Proof-theory

Two systems of increasing strength will be studied in this paper. Both are extensions of the system **F** presented in Åqvist [2, 3]. System **F** is axiomatized thus:

All truth functional tautologies	(PL)
S5-schemata for $\Box$ and $\Diamond$	(S5)
$P(B/A) \leftrightarrow \neg \bigcirc (\neg B/A)$	(DfP)

$$\bigcirc(B \to C/A) \to (\bigcirc(B/A) \to \bigcirc(C/A)) \tag{COK}$$

$$\bigcirc(B/A) \to \Box\bigcirc(B/A) \tag{Abs}$$

$$\Box A \to \bigcirc(A/B) \tag{Nec}$$

$$\Box(A \leftrightarrow B) \to (\bigcirc(C/A) \leftrightarrow \bigcirc(C/B)) \tag{Ext}$$

$$\bigcirc(A/A) \tag{Id}$$

$$\bigcirc(C/A \wedge B) \to \bigcirc(B \to C/A) \tag{Sh}$$

$$\Diamond A \to (\bigcirc(B/A) \to P(B/A)) \tag{D*}$$

$$\text{If } \vdash A \text{ and } \vdash A \to B \text{ then } \vdash B \tag{MP}$$

$$\text{If } \vdash A \text{ then } \vdash \Box A \tag{N}$$

A few comments on the axioms involving the deontic modalities might be in order. (DfP) introduces "P" as the dual of "$\bigcirc$" in the usual way. (COK) is the conditional analogue of the familiar distribution axiom K. (Abs) is the absoluteness axiom of Lewis [20], and reflects the fact that the ranking is not world-relative. (Nec) is the deontic counterpart of the familiar necessitation rule. (Ext) permits the replacement of equivalent sentences in the antecedent of deontic conditionals. (Id) and (Sh) are familiar from the literature on non-monotonic logic. (Id) is the deontic analogue of the identity principle. The question of whether this is a reasonable law for deontic conditionals has been much debated. A defense of (Id) can be found in Hansson [13] and Prakken and Sergot [28]. This line of defense is further discussed in Parent [27]. (Sh) is named after Shoham [31], who seems to have been the first to discuss it. It corresponds to the so-called "conditionalization" principle (also referred to as "the hard half of the deduction theorem"), which is part of Kraus and colleagues' system P for preferential consequence relations (see [18]).

F itself is described by Åqvist as providing a "partial" syntactic identification of (his own reconstruction of) DSDL2. It is said to be partial because, although (under the opt rule) the system is sound with respect to the class of models with $\succeq$ reflexive and opt-limited, the completeness problem has not been settled yet. Here no attempt will be made to tackle this problem.

The first system I shall consider in this paper is obtained by supplementing **F** with the so-called principle of cautious monotony from [18]:

$$(\bigcirc(B/A) \wedge \bigcirc(C/A)) \to \bigcirc(C/A \wedge B) \tag{CM}$$

I shall refer to this system as **F**+(CM). (CM) is not redundant at all. For it can be shown to be independent of the other axioms in **F**. This claim will be sustained further in Subsection 2.5.

The second system I shall consider in this paper is called **G** by Åqvist. It is obtained by supplementing **F** with this other axiom, due to Spohn [32]:

$$(P(B/A) \wedge \bigcirc(B \to C/A)) \to \bigcirc(C/A \wedge B) \tag{Sp}$$

The notions of theoremhood, deducibility and consistency are defined as usual. I write $\vdash A$ if A is provable, and $\Gamma \vdash A$ if A is derivable from Γ, where Γ is a set of wffs.

I shall make two comments on **G**. First, the latter is equivalently axiomatized if in the original axiomatization (Sp) is replaced with the law numbered as (12) in the list of DSDL3 validities given by Hansson [13, p. 397]. Lehmann and Magidor [19] subsequently coined the term "rational monotony" to refer to the latter law. This is the formula:

$$(P(B/A) \wedge \bigcirc(C/A)) \to \bigcirc(C/A \wedge B) \tag{RM}$$

It is straightforward to show that **F** contains the rule:

$$\text{If } \vdash B \to C \text{ then } \vdash \bigcirc(B/A) \to \bigcirc(C/A) \tag{RCOM}$$

Using (RCOM) one can easily show that (Sp) is deductively equivalent to (RM). The proof of the implication (Sp)⇒(RM) uses (RCOM) only. Assume $P(B/A)$ and $\bigcirc(C/A)$. From the latter, $\bigcirc(B \to C/A)$, by (RCOM). (Sp), then, yields $\bigcirc(C/A \wedge B)$ as required. The proof of the converse implication, (RM)⇒(Sp), uses (Id), (RCOM) and (COK). Assume $P(B/A)$ and $\bigcirc(B \to C/A)$. By (RM), $\bigcirc(B \to C/A \wedge B)$. By (Id) and (RCOM), $\bigcirc(B/A \wedge B)$ is a theorem. By (COK), $\bigcirc(C/A \wedge B)$ as required.

Second, (CM) is derivable in **G**, and thus **G** is a conservative extension of **F**+(CM). One might verify this claim by breaking the argument into cases. If we have $\Diamond A$, then (CM) follows from (RM), since (D*) allows us to weaken $\bigcirc(B/A)$ into $P(B/A)$. If we do not have $\Diamond A$, then (CM) follows from (Ext), because $\neg\Diamond A$ implies $\Box(A \leftrightarrow (A \wedge B))$.

For the completeness proofs, the following will come in handy.

Lemma 1 *Each of the following laws is provable in* **F**+*(CM):*

$$\bigcirc(B_1/A) \wedge ... \wedge \bigcirc(B_n/A) \to \bigcirc(B_1 \wedge ... \wedge B_n/A) \quad (n \geq 2) \tag{2}$$
$$\Diamond A \to \neg\bigcirc(\bot/A) \tag{3}$$
$$\bigcirc(B/A) \to \bigcirc(A \to B/A \vee C) \tag{4}$$
$$\bigcirc(C/A) \wedge \bigcirc(C/B) \to \bigcirc(C/A \vee B) \tag{5}$$
$$\bigcirc(B/A) \to (\bigcirc(C/A \wedge B) \to \bigcirc(C/A)) \tag{6}$$
$$\bigcirc(B/A) \wedge \bigcirc(A/B) \to (\bigcirc(C/A) \leftrightarrow \bigcirc(C/B)) \tag{7}$$
$$\bigcirc(C/A \vee C) \wedge \bigcirc(B/A) \to \bigcirc(A \to B/C) \tag{8}$$
$$\bigcirc(A/A \vee B) \wedge \bigcirc(B/B \vee C) \to \bigcirc(C \to B/A) \tag{9}$$

(3) is the law called COD by Chellas [6]. (6) is the principle of "cumulative transitivity", which is part of system C from Kraus *et al.* [18]. (7) is the axiom called CSO by Nute [24].

Proof The proofs of (2) and (3) are straightforward, and are omitted.

For (4), assume $\bigcirc(B/A)$. By (RCOM), $\bigcirc(A \to B/A)$. By (Ext), $\bigcirc(A \to B/(A \vee C) \wedge A)$. By (Sh), $\bigcirc(A \to (A \to B)/A \vee C)$. By (RCOM), $\bigcirc(A \to B/A \vee C)$.

For (5), assume $\bigcirc(C/A)$ and $\bigcirc(C/B)$. By law (4) in this Lemma, $\bigcirc(A \to C/A \vee B)$ and $\bigcirc(B \to C/A \vee B)$. By (2) and (RCOM), $\bigcirc((A \vee B) \to C/A \vee B)$. By (Id), $\bigcirc(A \vee B/A \vee B)$. By (2) and (RCOM), $\bigcirc(C/A \vee B)$.

For (6), assume $\bigcirc(B/A)$ and $\bigcirc(C/A \wedge B)$. By (Sh), $\bigcirc(B \to C/A)$. By (2) and (RCOM), $\bigcirc(C/A)$.

(7) follows from (CM) and (6) thus. Assume $\bigcirc(B/A)$, $\bigcirc(A/B)$, and $\bigcirc(C/A)$. From the first and third assumptions, by (CM) $\bigcirc(C/A \wedge B)$. This is equivalent to $\bigcirc(C/B \wedge A)$ by (Ext). From this together with the second assumption $\bigcirc(A/B)$, the conclusion $\bigcirc(C/B)$ follows, by (6). For the derivation of $\bigcirc(C/A)$ from $\bigcirc(C/B)$, the argument is similar.

For (8), assume $\bigcirc(C/A \vee C)$ and $\bigcirc(B/A)$. By (Ext), $\bigcirc(B/(A \vee C) \wedge A)$. By (Sh), $\bigcirc(A \to B/A \vee C)$. By (CM), $\bigcirc(A \to B/(A \vee C) \wedge C)$. By (Ext), $\bigcirc(A \to B/C)$.

For (9), assume $\bigcirc(A/A \vee B)$ and $\bigcirc(B/B \vee C)$. From the latter, one has $\bigcirc(B/(A \vee B \vee C) \wedge (B \vee C))$ by (Ext). Given (Sh), this implies $\bigcirc((B \vee C) \to B/A \vee B \vee C)$. From (RCOM), it follows that $\bigcirc(C \to B/A \vee B \vee C)$. By (Id), $\bigcirc(A \vee B/A \vee B)$. Using (RCOM), one also has $\bigcirc(A \vee B/B \vee C)$. Using law (5) in this Lemma and (Ext), one gets $\bigcirc(A \vee B/A \vee B \vee C)$.

By (CM), $\bigcirc(C \to B/(A \vee B \vee C) \wedge (A \vee B))$. This is equivalent to $\bigcirc(C \to B/A \vee B)$, by (Ext). But from this and the first assumption, using (CM), one obtains $\bigcirc(C \to B/(A \vee B) \wedge A)$, which is equivalent to $\bigcirc(C \to B/A)$, by (Ext). □

2.5 Independence

This subsection shows that (CM) is independent of the other axioms in system **F**.

Proposition 5 *Under the max rule,* **F** *is sound with respect to the class of models where* $\succeq$ *is reflexive and max-limited.*

Proof The proof is straightforward, and is omitted. Note that, under the max rule, max-limitedness validates (D*). □

Proposition 6 *Under the max rule, (CM) is falsifiable in a model where* $\succeq$ *is reflexive and max-limited.*

Proof The proof is provided by an example from Subsection 2.3. Let $W = \{x,y,z\}$, with $x \succ y$ and $y \succ z$ (reflexivity of the $\succeq$-type is left implicit). Put $V(p) = W$, $V(q) = \{x,z\}$, and $V(r) = \{x\}$. It is straightforward to show that $\succeq$ is max-limited. On the other hand, $\max_{\succeq}(\|p\|) = \{x\} = \|r\| \subseteq \|q\|$, so that $\bigcirc(r/p)$ and $\bigcirc(q/p)$ are true at each world. But $\max_{\succeq}(\|p \wedge q\|) = \{x,z\} \not\subseteq \|r\|$, so that $\bigcirc(r/p \wedge q)$ is false at each world. □

Corollary 1 *(CM) is not a theorem of* **F**.

Proof Given Proposition 5, if (CM) was a theorem of **F**, then (CM) would be valid in the class of models applying the max rule, in which $\succeq$ reflexive and max-limited. Proposition 6 shows that this cannot be. □

A similar argument can be given for the claim that (D*) is independent of (CM), even in the presence of the other axioms. Call $\mathbf{F}^-$ the system obtained by replacing, in **F**, (D*) with (CM).

Proposition 7 *Under the opt rule,* $\mathbf{F}^-$ *is sound with respect to the class of models where* $\succeq$ *is reflexive and condition (†) below is met.*

$$\textit{If } \mathrm{opt}_{\succeq}(\|A\|) \subseteq \|B\| \subseteq \|A\| \textit{ then } \mathrm{opt}_{\succeq}(\|B\|) \subseteq \mathrm{opt}_{\succeq}(\|A\|) \qquad (\dagger)$$

Proof The proof is straightforward, and is omitted. Note that, under the opt rule, (†) validates (CM). □

Proposition 8 *Under the opt rule, (D*) is falsifiable in a model where* $\succeq$ *is reflexive and (†) is met.*

Proof It suffices to reuse this other example from Subsection 2.3. Let $W = \{x,y\}$, with $x \succeq x$ and $y \succeq y$. Assume $\mathbb{P} = \{p,q\}$, and put $V(p) = V(q) = \{x,y\}$. It is not hard to establish that x and y are duplicate, i.e., for all wffs A, $x \models A$ iff $y \models A$.[6] From this, it follows that, for all wffs A, either $\|A\| = \emptyset$ or $\|A\| = W$, and so $\mathrm{opt}_{\succeq}(\|A\|) = \emptyset$ for all wffs A. Thus, on that model, (†) is vacuously met. And both x and y make $\Diamond p$, $\bigcirc(q/p)$, and $\bigcirc(\neg q/p)$ true, because $\|p\| \neq \emptyset$ whilst $\mathrm{opt}_{\succeq}(\|p\|) = \emptyset$. Hence, (D*) is falsified. □

Corollary 2 *(D*) is not a theorem of* $\mathbf{F}^-$.

Proof As per above. □

[6] The proof is by induction on A. (Note that the inductive hypothesis plays a role only for the cases in which A is $\neg B$, $B \wedge C$, $B \vee C$ or $B \to C$.)

3 Main Result

This section gives the chief result of this paper, Theorems 1 and 5 below. Henceforth, soundness and completeness will be understood in their strong version: they conjointly establish a match between the deducibility and the semantic consequence relations.

Theorem 1 (Soundness of F+(CM), max rule, partial order case) *Under the max rule,* **F**+*(CM) is sound with respect to the class of models in which $\succeq$ is reflexive and max-smooth.*

Proof It is enough to show the axioms are valid in the class of models under consideration, and the rules preserve validity over that class. I just consider the cases of (CM) and ($D^\star$):

- For (CM), consider a model in which $\succeq$ is max-smooth, and such that (i) $\max_{\succeq}(\|A\|) \subseteq \|B\|$, (ii) $\max_{\succeq}(\|A\|) \subseteq \|C\|$, and (iii) $\max_{\succeq}(\|A \wedge B\|) \not\subseteq \|C\|$. From (iii), there is some x such that $x \in \max_{\succeq}(\|A \wedge B\|)$ and $x \not\models C$. From (ii), $x \notin \max_{\succeq}(\|A\|)$, because $x \not\models C$. But $x \models A$. So, by max-smoothness, there is some y such that $y \succ x$ and $y \in \max_{\succeq}(\|A\|)$. Using (i), it follows that $y \models B$. But $y \models A \wedge B$, in contradiction with $x \in \max_{\succeq}(\|A \wedge B\|)$. This shows that, under the max rule, (CM) cannot be falsified in any model where $\succeq$ is reflexive and max-smooth, and thus it is valid in any such model.
- The validity of ($D^\star$) follows from max-smoothness along with Proposition 3 (a.i).

The argument for strong soundness is then as usual. Details are omitted. □

Completeness is shown using the canonical model method. Definitions and facts relative to maximal consistent sets of formulas are taken for granted. For the sake of brevity, the following abbreviations are used throughout the proof. MCS is short for “maximal consistent set”. The set of all the MCSs is denoted by Ω. Where x is a MCS, $\Box^{-1}(x)$ and x^A denote $\{B : \Box B \in x\}$ and $\{B : \bigcirc(B/A) \in x\}$, respectively. Given a MCS w, Ω_w is $\{x \in \Omega : \Box^{-1}(w) \subseteq x\}$.

I start by stating a few facts and lemmas. These will also help to establish a completeness result for **G**, even though the canonical model for **G** will slightly differ from the canonical model for **F**+(CM).

Fact 1 below shows that the relation $R \subseteq \Omega \times \Omega$, defined by putting xRy whenever $\Box^{-1}(x) \subseteq y$, is an equivalence relation. It divides Ω into mutually exclusive and collectively exhaustive non-empty subsets.

Fact 1 *(i) For all $x \in \Omega$, $\Box^{-1}(x) \neq \emptyset$; (ii) For all $x \in \Omega$, $\Box^{-1}(x) \subseteq x$; (iii) For all $x, y \in \Omega$, if $\Box^{-1}(x) \subseteq y$, then $\Box^{-1}(y) \subseteq x$; (iv) For all $x, y, z \in \Omega$, if $\Box^{-1}(x) \subseteq y$ and $\Box^{-1}(y) \subseteq z$, then $\Box^{-1}(x) \subseteq z$.*

Proof (i) follows from the necessitation axiom $\Box\top$.

(ii) follows from the T axiom $\Box A \to A$.

For (iii), let $x, y \in \Omega$ be such that $\Box^{-1}(x) \subseteq y$. Assume, to reach a contradiction, that $\Box A \in y$ and $A \notin x$. By maximality of x, $\neg A \in x$. Using the so-called *Brouwersche* axiom $A \to \Box\Diamond A$, it follows that $\Box\Diamond\neg A \in x$. By duality $\Box\neg\Box A \in x$. Since $\Box^{-1}(x) \subseteq y$, $\neg\Box A \in y$. So, by consistency of y, $\Box A \notin y$, a contradiction.

For (iv), let $x, y, z \in \Omega$ be such that $\Box^{-1}(x) \subseteq y$ and $\Box^{-1}(y) \subseteq z$. Let A be such that $\Box A \in x$. Using the law $\Box A \to \Box\Box A$, it follows that $\Box\Box A \in x$. Since $\Box^{-1}(x) \subseteq y$, $\Box A \in y$. Since $\Box^{-1}(y) \subseteq z$, $A \in z$ as required. □

Corollary 3 *For all $x, y, z \in \Omega$, if $\Box^{-1}(x) \subseteq y$ and $\Box^{-1}(x) \subseteq z$, then $\Box^{-1}(y) \subseteq z$.*

Proof This follows from parts (iii) and (iv) in Fact 1. □

Fact 2 *(i) For all $x \in \Omega$ and all wffs A, $x^A \neq \emptyset$; (ii) For all $x \in \Omega$ and all wffs A, $\Box^{-1}(x) \subseteq x^A$; (iii) For all $x \in \Omega_w$, $\bigcirc(B/A) \in w$ iff $\bigcirc(B/A) \in x$.*

Proof (i) follows from (Id).

For (ii), consider any B such that $\Box B \in x$. By (Nec), $\bigcirc(B/A) \in x$, and thus $B \in x^A$.

For (iii), consider any $x \in \Omega_w$. We have $\Box^{-1}(w) \subseteq x$. For the left-to-right direction, assume $\bigcirc(B/A) \in w$. By (Abs), $\Box\bigcirc(B/A) \in w$. So $\bigcirc(B/A) \in x$, since $\Box^{-1}(w) \subseteq x$. For the right-to-left direction, assume that $\bigcirc(B/A) \in x$. By (Abs), $\Box\bigcirc(B/A) \in x$. By Fact 1 (iii), $\Box^{-1}(x) \subseteq w$, so that $\bigcirc(B/A) \in w$ as required. □

For ease and conciseness of exposition, throughout the remainder of this paper $A \geq B$ is used as a shorthand for $\bigcirc(A/A \vee B) \in w$, where w is some MCS.

For the record I note the following.

Lemma 2 *(i) $A \vee B \geq A$; (ii) If $A \geq B$, $w^A \subseteq x$, and $B \in x$, then $w^B \subseteq x$; (iii) If $A \geq B \geq C$, $w^A \subseteq x$, and $C \in x$, then $w^B \subseteq x$.*

Proof (i) follows from (Id) and (Ext).

For (ii), assume $A \geq B$, $w^A \subseteq x$ and $B \in x$. Let $C \in w^B$. Then $\bigcirc(C/B) \in w$. Since $A \geq B$, $\bigcirc(A/A \vee B) \in w$. By Lemma 1 (8), $\bigcirc(B \to C/A) \in w$. Since $w^A \subseteq x$, $B \to C \in x$. Since $B \in x$, $C \in x$ as required.

For (iii), assume $A \geq B \geq C$, $w^A \subseteq x$, and $C \in x$. By Lemma 1 (9), $\bigcirc(C \to B/A) \in w$. But $w^A \subseteq x$, and thus $C \to B \in x$. Since $C \in x$, $B \in x$. By part (ii) of this Lemma, $w^B \subseteq x$, as required. □

Lemma 3 *(i) If $\Box A \notin w$, then $\Box^{-1}(w) \cup \{\neg A\}$ is consistent ; (ii) If $B \ngeq A$, then $w^{A \vee B} \cup \{\neg B\}$ is consistent.*

Proof The proof of (i) is standard, and is omitted.

For (ii), let $B \ngeq A$, and assume that $w^{A \vee B} \cup \{\neg B\}$ is not consistent. From the former, $\bigcirc(B/A \vee B) \notin w$. By Fact 2 (i), $w^{A \vee B} \neq \emptyset$. So, there are some C_1, ..., C_n ($n \geq 1$) in $w^{A \vee B}$, such that $\vdash (C_1 \wedge ... \wedge C_n) \to B$. For each C_i, $\bigcirc(C_i/A \vee B) \in w$; hence $\bigcirc(C_1/A \vee B) \wedge ... \wedge \bigcirc(C_n/A \vee B) \in w$, and thus $\bigcirc(C_1 \wedge ... \wedge C_n/A \vee B) \in w$ by Lemma 1 (2). Using (RCOM), it follows that $\bigcirc(B/A \vee B) \in w$, contrary to assumption. □

Lemma 4 *(i) If $A \in x$, where $x \in \Omega_w$, then w^A is consistent ; (ii) If $\bigcirc(B/A) \notin x$, where $x \in \Omega_w$, then $w^A \cup \{\neg B\}$ is consistent.*

Proof For (i), let $x \in \Omega_w$. Assume, to reach a contradiction, that $A \in x$, and that w^A is not consistent. By Fact 2 (i), $w^A \neq \emptyset$. So, for $n \geq 1$, there are some $C_1, .., C_n$ in w^A such that $\vdash (C_1 \wedge ... \wedge C_n) \to \bot$. For each C_i, $\bigcirc(C_i/A) \in w$; hence $\bigcirc(C_1/A) \wedge ... \wedge \bigcirc(C_n/A) \in w$, and thus $\bigcirc(C_1 \wedge ... \wedge C_n/A) \in w$ by Lemma 1 (2). Using (RCOM) it follows that $\bigcirc(\bot/A) \in w$. By Lemma 1 (3) and duality, $\Box\neg A \in w$, and thus $\neg A \in x$ since $x \in \Omega_w$. By consistency of x, $A \notin x$, contrary to assumption.

For (ii), let $x \in \Omega_w$. Assume, to reach a contradiction, that $\bigcirc(B/A) \notin x$, and that $w^A \cup \{\neg B\}$ is not consistent. By Fact 2 (i), $w^A \neq \emptyset$. So, for $n \geq 1$, there are some $C_1, .., C_n$ in w^A such that $\vdash (C_1 \wedge ... \wedge C_n) \to B$. A similar argument as before yields $\bigcirc(B/A) \in w$. But $x \in \Omega_w$. So, using Fact 2 (iii), it follows that $\bigcirc(B/A) \in x$. Contradiction. □

Now comes the completeness proof itself. It is centered around the construction of a point-generated canonical model. Let Γ be our "target" set of wffs: Γ is consistent, and its satisfiability in a model must be established. Let Γ_0 be some maximal extension of Γ. There are two cases to consider:

– $\Gamma_0^A \subseteq \Gamma_0$ for some A – principal case;
– $\Gamma_0^A \subseteq \Gamma_0$ for no A – limiting case.

I will work out these cases in turn, starting with the first one.

Definition 1 (Canonical model, principal case) Let w be a MCS such that $w^A \subseteq w$ for some A. The canonical model generated by (w,A) is the structure $M^{(w,A)} = (W, \succeq, V)$ defined by

(i) $W = \{(x,B) : x \in \Omega \;\&\; w^B \subseteq x\}$
(ii) $(x,B) \succeq (y,C)$ iff: either $C \not\geq B$ or $B \in y$
(iii) $V(p) = \{(x,B) \in W : p \in x\}$ for all $p \in \mathbb{P}$.

Lemma 5 *(i) $(w,A) \in W$; (ii) $W \neq \emptyset$.*

Proof (i) follows from Definition 1 (i) along with the assumption $w^A \subseteq w$.
(ii) follows from (i). □

The following applies.

Theorem 2 (Truth-lemma, principal case) *Let $M^{(w,A)} = (W, \succeq, V)$ be as in Definition 1. Then, for all $(x,E) \in W$ and all B, $(x,E) \models B$ iff $B \in x$.*

Proof By induction on B. I do the two cases: (a) $B = \Box C$ and (b) $B = \bigcirc(D/C)$.

Case (a.i). From right-to-left. Suppose $\Box C \in x$. We have $(x,E) \in W$. So, by Definition 1 (i), $w^E \subseteq x$. So $\Box^{-1}(w) \subseteq x$, by Fact 2 (ii). Let $(y,F) \in W$. By Definition 1 (i), $w^F \subseteq y$, and thus again $\Box^{-1}(w) \subseteq y$, by Fact 2 (ii). So $\Box^{-1}(x) \subseteq y$, by Corollary 3. Hence $C \in y$. The inductive hypothesis, then, yields $(y,F) \models C$, which suffices for $(x,E) \models \Box C$.

Case (a.ii). From left-to-right. Assume $\Box C \notin x$. By Lemma 3 (i), $\Box^{-1}(x) \cup \{\neg C\}$ is consistent, and can be extended to a MCS, call it y. We have $\neg C \in y$. As in case (a.i), $\Box^{-1}(w) \subseteq x$, by Fact 2 (ii). Also, $\Box^{-1}(x) \subseteq y$. By Fact 1 (iv), $\Box^{-1}(w) \subseteq y$, so that $y \in \Omega_w$. Using Lemma 4 (i), it follows that $w^{\neg C}$ is consistent, and can be extended to some MCS, call it z. Since $w^{\neg C} \subseteq z$, $(z, \neg C) \in W$ by Definition 1 (i). By (Id), $\neg C \in z$. By consistency of z, $C \notin z$. The inductive hypothesis, then, yields $(z, \neg C) \not\models C$, which suffices for $(x,E) \not\models \Box C$.

Case (b.i). From right-to-left. Suppose $\bigcirc(D/C) \in x$. Let $(y,F) \in \max_{\succeq}(\|C\|)$. We need to show that $(y,F) \models D$. Suppose, to reach a contradiction, that $F \not\geq C$. By Lemma 3 (ii), $w^{C \vee F} \cup \{\neg F\}$ is consistent, and can be extended to a MCS, call it z. Since $w^{C \vee F} \subseteq z$, $(z, C \vee F) \in W$. Furthermore, $C \vee F \in z$, by (Id). But $\neg F \in z$. So $C \in z$, and thus $(z, C \vee F) \models C$ by the inductive hypothesis. But $(y,F) \in W$, and thus $w^F \subseteq y$, by Definition 1 (i). So $F \in y$ by (Id), and thus $C \vee F \in y$. So $(z, C \vee F) \succeq (y,F)$, by Definition 1 (ii). On the other hand, by Lemma 2 (i), $C \vee F \geq F$, and $\neg F \in z$ implies $F \notin z$ by consistency of z. Therefore $(y,F) \not\succeq (z, C \vee F)$, by Definition 1 (ii). This contradicts the alleged maximality of (y,F) in $\|C\|$, and so we must conclude that $F \geq C$. But $(y,F) \models C$, and so the inductive hypothesis yields $C \in y$. By Lemma 2 (ii), $w^C \subseteq y$. Like in case (a.i), $\Box^{-1}(w) \subseteq x$. Hence $x \in \Omega_w$. So Fact 2 (iii) applies, yielding $\bigcirc(D/C) \in w$, so that $D \in w^C$. Hence $D \in y$. From this, $(y,F) \models D$ follows by the inductive hypothesis. Hence, $(x,E) \models \bigcirc(D/C)$.

Case (b.ii). From left-to-right. Suppose $\bigcirc(D/C) \notin x$. Like in case (a.i), $\Box^{-1}(w) \subseteq x$, i.e., $x \in \Omega_w$. So Lemma 4 (ii) can be used to conclude that $w^C \cup \{\neg D\}$ is consistent, and can be extended to a MCS, call it y. Since $w^C \subseteq y$, $(y,C) \in W$, by Definition 1 (i). By (Id), $C \in y$. By the inductive hypothesis, $(y,C) \models C$. Let $(z,F) \in W$ be such that $(z,F) \models C$. By the inductive hypothesis, $C \in z$, and thus $(y,C) \succeq (z,F)$ by Definition 1 (ii). So $(y,C) \in \max_{\succeq}(\|C\|)$. But $\neg D \in y$ implies $D \notin y$, by consistency of y. By the inductive hypothesis, $(y,C) \not\models D$, which suffices for $(x,E) \not\models \bigcirc(D/C)$. □

Proposition 9 *Let $M^{(w,A)} = (W, \succeq, V)$ be as in Definition 1. Then, $\succeq$ is reflexive.*

Proof Let $(x,B) \in W$. By Definition 1 (i), $w^B \subseteq x$. By (Id), $B \in x$, and thus $(x,B) \succeq (x,B)$ by Definition 1 (ii) as required. □

To show that max-smoothness holds, the following lemma will be useful.

Lemma 6 *For all $(x,B) \in W$, if $A \in x$ and $B \geq A$, then $(x,B) \in \max_{\succeq}(\|A\|)$.*

Proof Let $(x,B) \in W$. Assume $A \in x$ and $B \geq A$. From the first, $(x,B) \models A$, given Theorem 2. Assume, to reach a contradiction, that $(x,B) \notin \max_{\succeq}(\|A\|)$. There is, then, some $(y,C) \in W$ such that $(y,C) \models A$, $(y,C) \succeq (x,B)$ and $(x,B) \not\succeq (y,C)$. By Theorem 2, $A \in y$. By Definition 1 (ii), $(x,B) \not\succeq (y,C)$ implies $C \geq B$. So $C \geq B \geq A$. Furthermore, $w^C \subseteq y$, by Definition 1 (i). Lemma 2 (iii), then, yields $w^B \subseteq y$. So $B \in y$, by (Id). But $(x,B) \not\succeq (y,C)$ also implies $B \notin y$, by Definition 1 (ii). Contradiction. □

Proposition 10 *Let $M^{(w,A)} = (W, \succeq, V)$ be as in Definition 1. Then, $\succeq$ is max-smooth.*

Proof Let $(x,B) \in W$ be such that $(x,B) \models A$. By Definition 1 (i), $w^B \subseteq x$. On the other hand, $A \in x$ by Theorem 2. The argument is broken down into two cases:

Case 1: $B \geq A$. In this case the claim follows in virtue of Lemma 6.

Case 2: $B \not\geq A$. By Lemma 3 (ii), $w^{A \vee B} \cup \{\neg B\}$ is consistent, and can be extended to a MCS, call it y. By Definition 1 (i), $(y, A \vee B) \in W$. By (Id) $A \vee B \in y$, and thus $A \in y$ since $\neg B \in y$. So, by Theorem 2, $(y, A \vee B) \models A$. By Definition 1 (ii), $(y, A \vee B) \succeq (x,B)$, because $A \vee B \in x$. By Lemma 2 (i), $A \vee B \geq B$. Also $B \notin y$, by consistency of y. So $(x,B) \not\succeq (y, A \vee B)$ by Definition 1 (ii). Hence $(y, A \vee B) \succ (x,B)$. Let $(z,C) \models A$. By Theorem 2, $A \in z$, and thus $A \vee B \in z$. So, by Definition 1 (ii), $(y, A \vee B) \succeq (z,C)$, and hence $(y, A \vee B) \in \max_{\succeq}(\|A\|)$. □

Definition 2 is specifically tailored to the limiting case where $\Gamma_0^A \subseteq \Gamma_0$ for no A.

Definition 2 (Canonical model, limiting case) Let w be a MCS such that $w^A \subseteq w$ for no A. Take an arbitrarily chosen wff A. The canonical model generated by (w,A) is the structure $M^{(w,A)} = (W, \succeq, V)$ defined by

(i) $W = \widetilde{W} \cup \{(w,A)\}$, where $\widetilde{W} = \{(x,B) : x \in \Omega \ \& \ w^B \subseteq x\}$
(ii) $\succeq = \trianglerighteq \cup \{((w,A),(w,A))\} \cup \{(\alpha,(w,A)) : \alpha \in \widetilde{W}\}$ where $\trianglerighteq \subseteq \widetilde{W} \times \widetilde{W}$ is defined as in Definition 1, putting $(x,B) \trianglerighteq (y,C)$ iff either $C \not\geq B$ or $B \in y$
(iii) $V(p) = \{(x,B) \in W : p \in x\}$ for all $p \in \mathbb{P}$.

Theorem 3 (Truth-lemma, limiting case) *Let $M^{(w,A)}$ be as in Definition 2. Then, for all $(x,E) \in W$ and all B, $(x,E) \models B$ iff $B \in x$.*

Proof By induction on B. I focus on the two cases: (a) $B = \Box C$ and (b) $B = \bigcirc(D/C)$.

Case (a.i). From right-to-left. Suppose $\Box C \in x$. To show: $(y,F) \models C$ for all $(y,F) \in W$. The argument requires reasoning by cases:

- $(x,E) = (w,A)$. In that case $\Box C \in w$. By Fact 1 (ii), $C \in w$, and thus the inductive hypothesis yields $(w,A) \models C$. Consider any $(y,F) \in \widetilde{W}$. We have $w^F \subseteq y$. By Fact 2 (ii), $\Box^{-1}(w) \subseteq y$. So again $C \in y$, and thus the inductive hypothesis also yields $(y,F) \models C$.
- $(x,E) \in \widetilde{W}$. The same argument as that for the principal case, Theorem 2 above, yields $(y,F) \models C$ for any $(y,F) \in \widetilde{W}$. It remains to show that $(w,A) \models C$. This follows at once from Fact 2 (ii), Fact 1 (iii) and the inductive hypothesis.

Case (a.ii). From left-to-right. The proof is virtually the same as in the principal case, Theorem 2 above. If $(x,E) \in \widetilde{W}$, then the argument may be rerun without any required changes. The case where (x,E) is (w,A) is handled similarly with a notable simplification: the inclusion $\Box^{-1}(w) \subseteq y$ follows by mere construction of y.

Case (b.i). From right-to-left. The argument given for Theorem 2 appeals to the inclusion $w^F \subseteq y$. Such an inclusion may be established as follows. Suppose for *reductio* that $w^F \not\subseteq y$. Since $(y,F) \in W$, (y,F) is (w,A), Definition 2 (i). But $(y,F) \models C$, and thus the inductive hypothesis yields $C \in y$. By Fact 1 (ii), $w \in \Omega_w$, and so by Lemma 4 (i) w^C is consistent, and can be extended to some MCS, call it z. We have $(z,C) \in \widetilde{W} \subseteq W$. Also, $C \in z$, by (Id). Hence, by the inductive hypothesis, $(z,C) \models C$. Since (y,F) is (w,A), Definition 2 (ii) yields $(z,C) \succ (y,F)$. This contradicts the assumption $(y,F) \in \max_{\succeq}(||C||)$. So $w^F \subseteq y$. Once this noticed, one gets the desired conclusion by the same reasoning as in the principal case, except for when (x,E) is (w,A). In this case, the subordinate argument for $D \in y$ calls for simplification: $D \in y$ follows from $w^C \subseteq y$ and the opening assumption at once.

Case (b.ii). From left-to-right. The argument used for Theorem 2 still goes through. To apply Lemma 4 (ii), needed is the inclusion $\Box^{-1}(w) \subseteq x$. The latter still holds good when (x,E) is (w,A), given Fact 1 (ii). The world (y,C) is in $\widetilde{W}$, and so it is in W. The argument for $(y,C) \models C$ remains unchanged. The argument for $(y,C) \in \max_{\succeq}(\|C\|)$ may be rephrased thus. Let $(z,F) \in W$ be such that $(z,F) \models C$. If (z,F) is (w,A), then $(y,C) \succeq (z,F)$ by Definition 2 (ii). If (z,F) is not (w,A), then $(y,C) \trianglerighteq (z,F)$ (since $C \in z$, by the inductive hypothesis) and so $(y,C) \succeq (z,F)$, by Definition 2 (ii). The argument for $(y,C) \not\models D$ remains unchanged. $\Box$

Proposition 11 *Let $M^{(w,A)} = (W, \succeq, V)$ be as in Definition 2. Then, $\succeq$ is reflexive.*

Proof This follows from Proposition 9 and Definitions 2 (i) and (ii). $\Box$

Proposition 12 *Let $M^{(w,A)} = (W, \succeq, V)$ be as in Definition 2. Then, $\succeq$ is max-smooth.*

Proof Let $(x,C) \in W$ be such that $(x,C) \models B$. The proof is broken down into two cases.

Case 1: (x,C) is (w,A). By Theorem 3, $B \in w$. As with the argument regarding (w,A) under (b.i) above, Theorem 3, there is some $(y,B) \in \widetilde{W}$ with $B \in y$. By Theorem 3 again, $(y,B) \models B$. Since (x,C) is (w,A) and $(y,B) \in \widetilde{W}$, $(y,B) \succ (x,C)$ by Definition 2 (ii). Let $(z,D) \in W$ be such that $(z,D) \models B$. Either (a) (z,D) is (w,A) or (b) $(z,D) \in \widetilde{W}$. In case (a), $(y,B) \succeq (z,D)$ by Definition 2 (ii). In case (b), $B \in z$, by Theorem 3. So $(y,B) \trianglerighteq (z,D)$ and hence $(y,B) \succeq (z,D)$ by Definition 2 (ii). In both cases, $(y,B) \in \max_{\succeq}(\|B\|)$.

Case 2: $(x,C) \in \widetilde{W}$. Let $\|B\|^{\sim}$ stand for the restriction of $\|B\|$ to $\widetilde{W}$. $(x,C) \in \|B\|^{\sim}$. By Proposition 10, either (a) $(x,C) \in \max_{\trianglerighteq}(\|B\|^{\sim})$ or (b) there is $(y,D) \in \widetilde{W}$ such that $(y,D) \triangleright (x,C)$ and $(y,D) \in \max_{\trianglerighteq}(\|B\|^{\sim})$.

In case (a), $(x,C) \in \|B\|$. Let $(z,E) \in W$ be such that $(z,E) \in \|B\|$ and $(z,E) \succeq (x,C)$. When it is supposed that $(z,E) \succeq (x,C)$ and $(x,C) \in \widetilde{W}$, this entails that (z,E) is not (w,A) and that $(z,E) \trianglerighteq (x,C)$, by Definition 2 (ii). So $(z,E) \in \|B\|^{\sim}$. But $(x,C) \in \max_{\trianglerighteq}(\|B\|^{\sim})$, and thus $(x,C) \trianglerighteq (z,E)$, whence $(x,C) \succeq (z,E)$, by Definition 2 (ii). Thus, $(x,C) \in \max_{\succeq}(\|B\|)$.

In case (b), a similar argument yields that $(y,D) \succ (x,C)$ and $(y,D) \in \max_{\succeq}(\|B\|)$. $\Box$

Now, the chief result of this paper will come quickly.

Theorem 4 *Let Γ be a set of wffs that is consistent in $\mathbf{F}+(CM)$. Then, under the max rule, Γ is satisfiable in a model in which $\succeq$ is reflexive and max-smooth.*

Proof Let Γ be a consistent set of wffs. By Lindenbaum's lemma, Γ can be extended to some MCS, call it Γ_0. There are two cases to consider:

Case 1: $\Gamma_0^A \subseteq \Gamma_0$ for some A. In this case, the canonical model generated by (Γ_0, A), i.e., the structure $M^{(\Gamma_0, A)} = (W, \succeq, V)$, is as described in Definition 1. By Propositions 9 and 10, $\succeq$ meets the required conditions. By Lemma 5 (i), $(\Gamma_0, A) \in W$. Using Theorem 2 above, it follows that, for each wff B,

$$(\Gamma_0, A) \models B \text{ iff } B \in \Gamma_0$$

Since $\Gamma \subseteq \Gamma_0$, we have

$$(\Gamma_0, A) \models C \text{ for all } C \in \Gamma$$

as required.

Case 2: $\Gamma_0^A \subseteq \Gamma_0$ for no A. Let A be a fixed wff (its choice is arbitrary). The canonical model generated by (Γ_0, A) is as described in Definition 2. By Propositions 11 and 12, on that model $\succeq$ meets the required conditions. And, by construction, $(\Gamma_0, A) \in W$. Using Theorem 3, it follows that $(\Gamma_0, A) \models C$ for all $C \in \Gamma$, as required. □

Theorem 5 (Completeness of F+(CM), max rule, partial order case) *Under the max rule, $\mathbf{F}+(CM)$ is complete with respect to the class of models in which $\succeq$ is reflexive and max-smooth.*

Proof This follows from Theorem 4 in the usual way. Suppose that, under the max rule, $\Gamma \models A$. Then, under the max rule, $\Gamma \cup \{\neg A\}$ is not satisfiable in any model in which $\succeq$ is reflexive and max-smooth. Hence Theorem 4 gives $\Gamma \cup \{\neg A\} \vdash \perp$. By simple propositional manipulations, we get $\Gamma \vdash A$, as required. □

4 Auxiliary Results

From the result obtained in the previous section, and the constructions used in the proofs of the theorems, we may obtain a number of spin-off results. These are:

Theorem 6 (Soundness of F+(CM), max rule, total order case) *Under the max rule, **F**+(CM) is sound with respect to the class of models in which $\succeq$ is reflexive, max-smooth, and total.*

Proof The proof is virtually trivial, as none of the axioms depends on totalness. □

Theorem 7 (Completeness of F+(CM), max rule, total order case) *Under the max rule, $\mathbf{F}+(CM)$ is complete with respect to the class of models in which $\succeq$ is reflexive, max-smooth, and total.*

Proof All that is required to show is that, in the canonical model of Theorem 4, $\succeq$ satisfies the property of totalness. Let (x, B) and (y, C) be in W. Assume, to reach a contradiction, that $(x, B) \not\succeq (y, C)$ and $(y, C) \not\succeq (x, B)$.

Case 1: $\Gamma_0^A \subseteq \Gamma_0$ for some A. In that case, the canonical model generated by (Γ_0, A) is as in Definition 1. By Definition 1 (ii), $C \geq B$, $B \geq C$, $B \notin y$, and $C \notin x$. From the first two,

$\bigcirc(C/B \vee C) \in w$ and $\bigcirc(B/B \vee C) \in w$. By Lemma 1 (9), $\bigcirc(C \to B/C) \in w$. By (Id) and (COK), $\bigcirc(B/C) \in w$. But $w^C \subseteq y$, and thus $B \in y$. Contradiction.

Case 2: $\Gamma_0^A \subseteq \Gamma_0$ for no A. In that case, the canonical model generated by (Γ_0, A) is as in Definition 2. When it is supposed, for *reductio*, that $(x,B) \not\succeq (y,C)$ and $(y,C) \not\succeq (x,B)$, that entails that neither is (w,A), by the definition of $\succeq$. So (x,B) and (y,C) are both in $\widetilde{W}$, and the claim follows for the same reason as in case 1.

An analogue of Theorem 4 immediately follows: if a set Γ of wffs is consistent in $\mathbf{F} + (CM)$, then it is satisfiable in a model applying the max rule, in which $\succeq$ is reflexive, max-smooth, and total. The argument then recapitulates the proof of Theorem 5. □

The methods of this paper may be applied to obtain a determination result for **G** under the max rule. Before showing this, it may be helpful to have the following lemma:

Lemma 7 *The following holds in* $\mathbf{F} + (CM)$*, and thus in* **G***:*

$$\vdash \bigcirc(\neg B/A \vee B) \to \bigcirc(A/A \vee B) \tag{10}$$

$$\vdash \bigcirc(A/A \vee B) \wedge \bigcirc(C/A \vee B) \to \bigcirc(C/A) \tag{11}$$

$$\text{If } A \geq B, \text{ then } w^A = w^{A \vee B} \tag{12}$$

Proof (10) follows from (Id), (RCOM) and Lemma 1 (2).
(11) follows from (CM) and (Ext).
(12) follows from Lemma 1 (7), (Id) and (RCOM). □

Theorem 8 (Soundness of G, max rule) *Under the max rule,* **G** *is sound with respect to the class of models in which $\succeq$ is reflexive, transitive, total, and max-smooth (or max-limited).*

Proof Given totalness and transitivity, max-smoothness and max-limitedness are equivalent, by Proposition 3 (a). Now, it suffices to show that, under the max rule, (Sp) is valid as long as $\succeq$ is required be transitive and total. Consider a model in which $\succeq$ is both transitive and total. Assume that (i) $\max_{\succeq}(\|A\|) \subseteq \|B \to C\|$, (ii) $\max_{\succeq}(\|A\|) \cap \|B\| \neq \emptyset$, and (iii) $\max_{\succeq}(\|A \wedge B\|) \not\subseteq \|C\|$. From (iii), there is some x such that $x \in \max_{\succeq}(\|A \wedge B\|)$ and $x \not\models C$. From (i), $x \notin \max_{\succeq}(\|A\|)$, because $x \not\models B \to C$. But $x \models A$. So there is some $y \models A$ with $y \succ x$. From (ii), there is also some z such that $z \in \max_{\succeq}(\|A\|)$ and $z \models B$. Totalness of $\succeq$ guarantees that $z \succeq y$. And $y \succ x$ implies $y \succeq x$ and $x \not\succeq y$. From $y \succeq x$ and $z \succeq y$, one gets $z \succeq x$ by transitivity. From $x \not\succeq y$ and $z \succeq y$, one also gets $x \not\succeq z$, by transitivity again. But $z \models A \wedge B$, and so $x \notin \max_{\succeq}(\|A \wedge B\|)$, contrary to assumption. This shows that (Sp) cannot be falsified in any model in which $\succeq$ is both transitive and total. □

As regards completeness, the canonical model previously introduced would not do the required job, because there is no guarantee that, on that model, $\succeq$ is transitive when (Sp) is added as an axiom schema. However, the proposed construction can be adapted to suit our needs.

Theorem 9 (Completeness of G, max rule) *Under the max rule,* **G** *is complete with respect to the class of models in which $\succeq$ is reflexive, transitive, total, and max-smooth (or max-limited).*

Proof Define the canonical model generated by (w,A) as the structure $M^{(w,A)} = (W, \succeq, V)$, where

- $W = \{(x,B) : x \in \Omega_w \,\&\, B \in \mathscr{L}\}$
- $(x,B) \succeq (y,C)$ iff
  either $w^C \not\subseteq y$ or ($w^B \subseteq x$ and ($w^C \subseteq x$ or $C \not\geq B$ or $B \in y$))
- $V(p) = \{(x,B) \in W : p \in x\}$ for all $p \in \mathbb{P}$.

Facts 1 and 2, and Lemmas 2 through 4 are still applicable even though the canonical model being developed is slightly different from that for $\mathbf{F} + (CM)$.

The reader can easily verify that the truth-lemma holds, and that $\succeq$ is total, thereby reflexive. The proof of transitivity of $\succeq$ is given in full. It highlights the importance of (Sp).

Let (x,B), (y,C) and (z,D) be in W. Suppose $(x,B) \not\succeq (y,C)$. Then, $w^C \subseteq y$, and either (a) $w^B \not\subseteq x$ or (b) $w^C \not\subseteq x$, $C \geq B$ and $B \notin y$. Suppose (a) holds. In that case, either $(z,D) \not\succeq (y,C)$ or $(x,B) \not\succeq (z,D)$. This is because either $w^D \not\subseteq z$ or $w^D \subseteq z$. The case where (b) holds needs a little more care. By Lemma 7 (12), $w^C = w^{B\vee C}$, and thus $w^{B\vee C} \not\subseteq x$. It, then, follows that $\bigcirc(\neg B/B\vee C) \in w$. Otherwise, given (DfP), (Ext), (RCOM) and (Sp), we would get $w^{B\vee C} \subseteq w^B$, from which $w^{B\vee C} \subseteq x$ would follow. If $w^D \not\subseteq z$, then $(z,D) \not\succeq (y,C)$. So suppose $w^D \subseteq z$.

Case 1: $\bigcirc(\neg(B\vee D)/B\vee C\vee D) \in w$. By (RCOM), $\bigcirc(\neg B/B\vee C\vee D) \in w$ and $\bigcirc(\neg D/B\vee C\vee D) \in w$. From the first, $\bigcirc(C\vee D/B\vee C\vee D) \in w$ by Lemma 7 (10). From the last two, $\bigcirc(\neg D/C\vee D) \in w$ by Lemma 7 (11) and (Ext). Re-using (10) in this Lemma, we get $\bigcirc(C/C\vee D) \in w$, viz. $C \geq D$. Re-using (11) in this Lemma, we get $\bigcirc(\neg D/C) \in w$. Thus, $\neg D \in y$, and so $D \notin y$ by consistency of y. By contrast, $D \in z$ by (Id), and so $\neg D \notin z$ by consistency of z. Hence, $w^C \not\subseteq z$. Therefore, $(z,D) \not\succeq (y,C)$.

Case 2: $\bigcirc(\neg(B\vee D)/B\vee C\vee D) \notin w$. By maximality of w and (DfP), $P(B\vee D/B\vee C\vee D) \in w$. By Lemma 1 (4), $\bigcirc((B\vee C) \to \neg B/B\vee C\vee D) \in w$. Using (RCOM), it follows that $\bigcirc((B\vee D) \to ((B\vee C) \to \neg B)/B\vee C\vee D) \in w$. Using (Sp) and (Ext), we then get $\bigcirc((B\vee C) \to \neg B/B\vee D) \in w$. (RCOM) yields $\bigcirc(\neg B/B\vee D) \in w$. By Lemma 7 (10), $\bigcirc(D/B\vee D) \in w$, viz $D \geq B$. A similar argument as before yields $w^D \not\subseteq x$ and $B \notin z$, which suffices for $(x,B) \not\succeq (z,D)$.

It is readily checked that $\succeq$ is max-limited, and hence max-smooth. □

The results for optimality are now presented all together. The partial and total order cases are run in parallel.

Theorem 10 (Soundness of F+(CM), opt rule) *Under the opt rule,* **F**+*(CM) is sound with respect to (i) the class of models in which $\succeq$ is reflexive and opt-smooth, and (ii) the class of models in which $\succeq$ is reflexive, opt-smooth and total.*

Proof For (i), suffice it to say that opt-smoothness has the effect of validating both (CM) and (D*), given Proposition 3 (b.i). For (ii), it is enough to observe that, under the opt rule, none of the postulates of **F**+(CM) depends on the assumption of totalness. □

Theorem 11 (Completeness of F+(CM), opt rule) *Under the opt rule,* **F**+*(CM) is complete with respect to (i) the class of models in which $\succeq$ is reflexive and opt-smooth, and (ii) the class of models in which $\succeq$ is reflexive, opt-smooth and total.*

Proof For (ii), it suffices to observe that, in the canonical model for **F**+(CM), $\succeq$ is total. Thus, by Proposition 1, on that model, the truth-lemma, Theorems 2 and 3, also holds when deontic formulas are evaluated using the opt rule. Furthermore, by Proposition 2 (b.ii), on that model $\succeq$ is opt-smooth. Hence an analogue of Theorem 4 applies again. That is, if Γ is consistent in **F**+(CM), then it is satisfiable in a model applying the opt rule, in which $\succeq$ is reflexive, opt-smooth and total.

(i) follows from the above and the fact that deleting a condition on $\succeq$ does not affect the satisfiability of a set of wffs. That is, if Γ is satisfiable in a model applying the opt rule, in which $\succeq$ is reflexive, opt-smooth and total, then *a fortiori* Γ is satisfiable in a model applying the opt rule, in which $\succeq$ is reflexive and opt-smooth. $\square$

Theorem 12 (Soundness of G, opt rule) *Under the opt rule,* **G** *is sound with respect to (i) the class of models in which $\succeq$ is reflexive, transitive and opt-limited (or opt-smooth), and (ii) the class of models in which $\succeq$ is reflexive, transitive, opt-limited (or opt-smooth) and total.*

Proof For (i), observe first that, given transitivity, opt-limitedness and opt-smoothness are equivalent, Proposition 3 (b). Next, it suffices to verify that, under the opt rule, the validity of (Sp) depends on transitivity only. Consider a model where $\succeq$ is transitive, and such that (i) $\text{opt}_{\succeq}(\|A\|) \subseteq \|B \to C\|$, (ii) $\text{opt}_{\succeq}(\|A\|) \cap \|B\| \neq \emptyset$ and (iii) $\text{opt}_{\succeq}(\|A \wedge B\|) \not\subseteq \|C\|$. From (iii), there is some x such that $x \in \text{opt}_{\succeq}(\|A \wedge B\|)$ and $x \not\models C$. From (i), $x \notin \text{opt}_{\succeq}(\|A\|)$, because $x \not\models B \to C$. But $x \models A$. So there is some $y \models A$ with $x \not\succeq y$. From (ii), there is also some z such that $z \in \text{opt}_{\succeq}(\|A\|)$ and $z \models B$. We have $z \models A \wedge B$, and so $x \succeq z$. Also, $z \succeq y$. By transitivity, we get $x \succeq y$ - a contradiction.

(ii) holds for the same reason as before. $\square$

Theorem 13 (Completeness of G, opt rule) *Under the opt rule,* **G** *is complete with respect to (i) the class of models in which $\succeq$ is reflexive, transitive and opt-limited (or opt-smooth), and (ii) the class of models in which $\succeq$ is reflexive, transitive, opt-limited (or opt-smooth) and total.*

Proof (ii) follows from Theorem 9 along with Propositions 1, 2 (b.ii) and 3 (b.i). Consider the canonical model for **G** in the proof of Theorem 9. On that model, $\succeq$ is total. So, by Proposition 1, the truth-lemma holds good when deontic formulas are interpreted using the opt rule. On that model, $\succeq$ is max-smooth. So, by Proposition 2 (b.ii), $\succeq$ is also opt-smooth, and so it is opt-limited, Proposition 3 (b.i).

(i) follows from (ii). $\square$

An alternative proof of Theorem 13, based on a simpler canonical model construction, may be found in Parent [25]. There is a flaw in the argument establishing that in the canonical model (as defined there) $\succeq$ is total. It can be repaired, by using Proposition 13 below. It is the key to understanding totalness under the opt rule.

Lemma 8 *Let $M = (W, \succeq, V)$ be a model where $\succeq$ is reflexive, transitive and opt-limited. For all $x, y \in W$, if $x \in \text{opt}_{\succeq}(\|A\|)$ and $y \in \text{opt}_{\succeq}(\|B\|)$, then either $x \in \text{opt}_{\succeq}(\|A \vee B\|)$ or $y \in \text{opt}_{\succeq}(\|A \vee B\|)$.*

Proof Let $x \in \text{opt}_{\succeq}(\|A\|)$ and $y \in \text{opt}_{\succeq}(\|B\|)$. Suppose, to reach a contradiction, that $x \notin \text{opt}_{\succeq}(\|A \vee B\|)$ and $y \notin \text{opt}_{\succeq}(\|A \vee B\|)$. Since $x \models A \vee B$ and $y \models A \vee B$, $x \not\succeq x'$ and $y \not\succeq y'$ for some x' and y' such that $x' \models A \vee B$ and $y' \models A \vee B$. By opt-limitedness, there is some z such that $z \in \text{opt}_{\succeq}(\|A \vee B\|)$. So $z \succeq x'$ and $z \succeq y'$. Suppose $z \models A$. In that case, $x \succeq z$. By transitivity, one gets the contradiction that $x \succeq x'$. The argument for the case where $z \models B$ is similar. $\square$

Lemma 9 *Let $M = (W, \succeq, V)$ be a model where $\succeq$ is reflexive, transitive and opt-limited. For all $x, y \in W$, if $x \in \text{opt}_{\succeq}(\|A\|)$ and $y \in \text{opt}_{\succeq}(\|B\|)$, then either $x \succeq y$ or $y \succeq x$.*

Proof Immediate from Lemma 8. □

From this, we get

Proposition 13 *For every model $M = (W, \succeq, V)$ with $\succeq$ reflexive, transitive and opt-limited, there is a model $M^+ = (W, \succeq^+, V)$ (with W and V the same as in M) such that:*

1. *Under the opt rule, M^+ is equivalent to M, i.e., for all A and all $x \in W$,*
$$M, x \models A \text{ iff } M^+, x \models A;$$
2. *$\succeq^+$ is reflexive, transitive, opt-limited and total.*

Proof Starting with $M = (W, \succeq, V)$, define the relation $\succeq^+$ thus:

a) Let $\Pi = \{x \in W : (\exists A)(x \in \text{opt}_{\succeq}(\|A\|^M))\}$
b) For all $x, y \in W$:
   i) If $x, y \in \Pi$, then: $x \succeq^+ y$ iff $x \succeq y$;
   ii) If $x \in \Pi$ and $y \notin \Pi$, then $x \succ^+ y$;
   iii) If $x \notin \Pi$ and $y \notin \Pi$, then $x \succeq^+ y$ and $y \succeq^+ x$.

Given Lemma 9, it is straightforward to show that $\succeq^+$ meets the required constraints, and that a world in both models satisfies exactly the same sentences. Details are omitted. □

By Proposition 3 (b), Proposition 13 still holds if opt-limitedness is replaced with opt-smoothness.

I end with a few words on how the present framework, which is in keeping with Åqvist's investigations, compares to Hansson's one.

The language goes beyond Hansson's one with respect to including alethic modalities, mixed formulas and iterated deontic modalities. The use of mixed formulas and alethic modalities seems desirable. In particular, once the language has been supplemented with $\Box$, a seemingly plausible form of detachment is validated by the semantics. The law is known as "strong factual detachment": from $\Box A$ and $\bigcirc(B/A)$, infer $\bigcirc B$. Here the "$\Box$" modality is usually interpreted as a settledness operator.[7] Van Eck asked, "How can we take seriously a conditional obligation if it cannot, by way of detachment, lead to an unconditional obligation?" [34, p. 263] The issue of iterated deontic modalities is more complex, and I shall not take a stance on it. I shall just point out that, with S5 for the alethic modalities and the "bridge" principles (Abs) and (Nec), then (4) $\bigcirc A \to \bigcirc\bigcirc A$ and (5) $PA \to \bigcirc PA$ are derivable. So, the (monadic) deontic logic will not be the usual SDL = KD, but rather KD45.[8]

More significant, perhaps, than what is included in the language itself, is the fact that the models used to interpret the language as given here are quite different from the models Hansson describes. We see this in the contrast between the Kripke-style possible worlds semantics applied here and Hansson's own semantics based on alternative valuations for atomic formulas. The possible worlds approach may be dictated by the inclusion of alethic modalities and iterated deontic modalities. The difference, though, has consequences for the class of validities. In particular, Hansson's semantics has the law $\models \bigcirc(A/p) \to P(A/p)$. The latter formula is not valid in the Åqvist possible world semantics. For this reason, Theorems 9 and 13 do not apply to Spohn's axiomatization of DSDL3. It could be that the methods

[7] The use of a settledness operator is in line with Hansson's own interpretation of circumstances in [13, §13]. For more on strong factual detachment, see [28, 27].

[8] I owe this observation to the referee. A proof of completeness of SDL with respect to various classes of preference models applying the so-called Danielsson-type evaluation rule may be found in Goble [7, §1].

used in the proofs may be adapted to yield a strong completeness result with respect to Hansson's valuation semantics. But this remains to be seen. For DSDL2, there is little prospect of success, because Hansson uses the limit assumption in the form of max-limitedness. Thus, (CM) is not available anymore.

5 Wrap-up

Let us recap the main highlights. This paper has reported completeness results for dyadic deontic logics in the tradition of Hansson's systems. Taken together, they shed light on the question of how a semantics interpreting deontic formulas in terms of maximality compares with one interpreting them in terms of optimality. Four versions of the limit assumption have been disentangled, and compared. If one has a closer look at Table 1 below, which summarizes the results established in this paper, one can see that the contrast between the two approaches is not as significant as one could expect. The leftmost column shows the constraints placed on $\succeq$. The other two columns show the corresponding systems, the middle column for models applying the max rule, and the rightmost one for models applying the opt rule. In order to facilitate comparison of results, the limit assumption is phrased as smoothness throughout. It is understood that smoothness is defined for max in the max column, and for opt in the opt column. The top row considers two cases, separated by a dashed line: the case where $\succeq$ is reflexive and smooth, and the case where it is also total. The bottom row should be understood the same way.

<table>
<tr><th></th><th>max</th><th>opt</th></tr>
<tr><td>reflexivity of $\succeq$
$+$ smoothness</td><td rowspan="2">F+(CM)</td><td rowspan="2">F+(CM)</td></tr>
<tr><td>$+$ totalness</td></tr>
<tr><td>reflexivity
$+$ smoothness
$+$ transitivity</td><td>?</td><td rowspan="2">G</td></tr>
<tr><td>$+$ totalness</td><td>G</td></tr>
</table>

Table 1: Soundness and completeness results

The chief result of this paper is shown in the top row. It concerns the non-necessarily transitive case. Under both the max and opt rules, **F**+(CM) is sound and complete with respect to the class of models in which $\succeq$ reflexive and smooth, and the class of those in which it is also total. For the maximality account, this has been shown from first principles. From this first result, an analogue one has followed quite naturally for the optimality account. It may come as a surprise that, in the partial order case, the axiomatic characterization remains the same. It is also striking that, whatever type of evaluation rule is used, the assumption of totalness has no import at all. This was already reckoned by Spohn [32, §4.2], but for DSDL3, and under the opt rule only.

The bottom row shows what happens when transitivity of $\succeq$ comes into the picture. First, under both the max and opt rules, **G** is sound and complete with respect to the class of models in which $\succeq$ is reflexive, transitive, total and smooth (or, equivalently, limited). This determination result was proved directly under the max rule. In the presence of totalness, the latter result carried over to the opt rule. Here, a first difference between maximality and optimality emerges: under the max rule, the distinctive axiom of **G**, (Sp), requires both transitivity and totalness, while under the opt rule it requires transitivity only. This may seem a rather small point, but it has a non-trivial consequence: under the opt rule, totalness alone has no import at all again. Thus, under the opt rule, **G** is also sound and complete with respect to the class of models in which $\succeq$ is reflexive, transitive, and opt-smooth (or, equivalently, opt-limited). There is nothing similar for **G** under the max rule.

What are the lessons from all this? Well, there are significant cases where, under both accounts, the same system is sound and complete with respect to the intended modelling, given analogous properties for the betterness relation. Thus, the choice between maximality and optimality has no effect on the logic of "ought". Such a claim is not as paradoxical as might appear at first sight. It should just be taken to mean that the set of inference rules that can conceivably be used for assessing ordinary arguments involving ought-sentences remains the same − whatever notion of best is used.

Admittedly, Table 1 leaves out a number of possibilities − their study is a topic for future research. To take one first example, it would be interesting to know if a similar point can be made about the class of models in which $\succeq$ is reflexive and limited.[9] However, the above determination results for **F**+(CM) do not carry over to this class of models, because (CM) goes away under both rules. It was mentioned that, under the opt rule, **F** is sound with respect to the class of models in which $\succeq$ is reflexive and opt-limited. It is not difficult to see that this also holds under the max rule, with opt-limitedness replaced with max-limitedness. However, under both rules, the completeness problem of **F** has not been settled yet.[10]

The question of how to axiomatize the class of models applying the max rule, in which $\succeq$ is reflexive, max-smooth and transitive, is an open problem too. In the table, this is indicated by a question mark. Here, max-smoothness is no longer equivalent to max-limitedness. The question of whether (under the max rule) transitivity alone has a syntactical counterpart has not been investigated in this paper. The answer to this question is not straightforward, because the candidate formula must be falsifiable in a model with $\succeq$ reflexive and max-smooth. The main difficulty is to show that, in the model in question, $\succeq$ still satisfies max-smoothness. In the absence of transitivity, finiteness of the universe is no longer a guarantee of max-smoothness.[11]

We will not get the entire picture of how maximality compares with optimality until these questions, among others, are answered. But at least I hope to have made a first significant step towards clarifying this issue.

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[9] Their "max" variant comes close to Hansson's own picture of DSDL2.

[10] In Parent [26], the class of models applying the opt rule, with $\succeq$ reflexive and opt-limited, is axiomatized using an alternative language, which has the unary modal operator "Q" ("ideally", ...) as main building block. The proof of completeness given there makes an essential use of the axiom $QA \wedge QB \to Q(A \vee B)$, which corresponds to Sen's property γ (see [30]) in modal logic notation. Although the axiom remains valid under the max rule, it has no obvious counterpart in terms of $\bigcirc(-/-)$.

[11] Cf. Proposition 4, and the example immediately before.

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