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Cumulativity, Identity and Time in Deontic Logic*

Xavier Parent
Department of Philosophy
University of Aix-Marseille I
Aix-en-Provence, France
xp@up.univ-aix.fr

Abstract. We first analyse the future-orientated account of conditional obligation as put forth by David Makinson. We show that the introduction of the futurity dimension is not indispensable in the elimination of the identity principle. This leads us to a more general observation. Most systems of dyadic deontic logic either accept both the identity principle and the cumulativity condition, or exclude both. We then develop an account of defeasible conditional obligation based on the "and next" operator, as a solution to this dilemma.

Keywords: Preferential models, defeasible deontic logic, futurity, "next" operator

1. Introduction

Preferential structures were first introduced in deontic logic by Hansson [15], in order to give a semantical analysis of contrary-to-duty (or secondary) obligations that tell us what comes into force when some other (primary) obligations are violated. A number of researchers have followed Hansson's suggestion, providing a more comprehensive investigation of the treatment of the contrary-to-duty obligations within a preference-based approach. It is not the purpose of this paper to evaluate such treatment, as this has already been extensively discussed in the literature [27, 34, 19, 35, 30, 6, 31, 37, 38].

In the following, we focus on another motivation for using preferential structures in the analysis of obligation sentences, as put forward by Alchourrón [1]. His idea is to use such structures so as to make the expression of defeasible conditional obligation possible. A defeasible obligation, he says, is one that may admit exceptions. But, may it naively (and rightly) be remarked, can an obligation be treated as having exceptions? If so treated, is it still an obligation? To

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answer this question, Carlos E. Alchourrón refers to Ross's concept of *prima facie* norms. The fact that a conditional obligation is defeasible and may admit exceptions simply means that, on a particular occasion, it may conflict with and be overridden by another stronger obligation (while remaining in force, as a prima facie obligation). Then, it may be asked what justifies the use of preferential semantics in the analysis of defeasible (or prima facie) obligation. Briefly speaking, it is the fact that such a semantics rejects as invalid the deontic analogue of two laws, modus-ponens and monotony, the failure of which constitutes the main formal feature expected for defeasible statements. Expressed in terms of the notation introduced by von Wright [39], where $\bigcirc(B/A)$ is for "the conditional obligation of B, given A", this means that the following two principles both fail: $\bigcirc(B/A)$ and A imply $\bigcirc B$ (deontic modus-ponens); $\bigcirc(B/A)$ entails $\bigcirc(B/A \wedge C)$ (deontic monotony).

There are close similarities between the logic of defeasible conditional obligations and so-called non-monotonic logics, as developed since the early eigthies in the context of logics for artificial intelligence. The contrasts which nonetheless exist render the comparison between the two fruitful. This article is about a "dissymetry" which can be found at the syntactic level between the properties of the operators in question. As first remarked by Gabbay [13], it is noticeable that any non-monotonic inference relation $|\!\sim$ should simultaneously satisfy the rule of reflexivity (or identity) and the principle of cumulativity. We recall here that there are two sides to the latter. One is cut (or cumulative transitivity): from $A |\!\sim B$ and $A \wedge B |\!\sim C$ infer $A |\!\sim C$. The other is cautious monotony (or cumulative monotony): from $A |\!\sim B$ and $A |\!\sim C$ infer $A \wedge B |\!\sim C$. Of these core conditions for $|\!\sim$, only the last two are conceptually required to hold for $\bigcirc(/)$. The deontic version of identity, $\bigcirc(A/A)$, is clearly counterintuitive, as some have observed [8, 32, 1, 22]. Other logicians, such as Nute and Yu [28, 29], have noticed that the deontic counterpart of cut has strong intuitive support, and this line of thought is likewise applicable to cautious monotony.

Within the framework of preference-based semantics, two main approaches have dominated the study of ought-sentences. Some haved treated the conditional obligation operator as a primitive dyadic construction (to which they have associated a comparative goodness relation). This is the approach initiated by Hansson [15]. Others have defined the dyadic obligation operator in terms of the standard (monadic) obligation operator together with a Lewis-type conditional connective. This approach apparently originated with Chellas [8] and Mott [27]. It is our contention here that both accounts are problematic. In the former, the cumulativity condition and the identity principle both hold. In the latter, they both fail. Our paper is an attempt to resolve the dilemma in which we are left by such mutually exclusive accounts. The main issue we address is whether a form of cumulativity can be preserved for obligations, whilst continuing to exclude the principle of identity. We present a solution to this difficulty, and investigate a new account of defeasible conditional obligation, based on von Wright's logic of "and next" [40, 3].

Our paper explores another important topic: the relation of conditional obligation with time. Since the seventies, this issue has generated a substantial body of literature [3, 7, 32, 9, 12, 19, 33, 38, 5]. It is noticeable that, in its earliest form as developed by Hansson, preference-based semantics had no apparatus for handling futurity – the consideration of which is generally a prime concern in the study of norms. Among the attempts to capture this aspect of time, Makinson's study [22] is of particular interest. In our paper, we analyse his proposal in detail, and show that the introduction of the futurity dimension is not indispensable to eliminate the principle of identity.

This article is organized as follows. Section 2 fleshes out the aforementionned dilemma, through a careful examination of David Makinson's attempt to capture the future character. In subsection 2.1 we introduce the new evaluation rule he proposes for the dyadic obligation operator. In subsection 2.2 our concern is to clarify how the proposed construction compares with that used by Hansson. In subsections 2.3 and 2.4 we investigate the question of whether the futurity dimension can help to resolve the "tension" or dilemma we started with. We argue that it cannot. So, in section 3, we go one step further in our investigation of the interaction between conditional obligation and time. In subsection 3.1, we give an account of defeasible conditional obligation in which the condition is required to be temporally prior to the content of obligation. Next, in subsection 3.2, we consider how this new account deals with the above problem.

2. A Problem with Identity and Cumulativity

In this section, we analyse David Makinson's account of defeasible conditional obligation, as put forward at the end of [22]. There, the author suggests that an adequate treatment of conditional obligation should both reject the identity principle and capture the future character. In the following, we investigate the relationship between the two requirements. This investigation will pave the way for a more general observation about current theories of conditional obligation.

There is one remark to make, before we introduce David Makinson's account. Some have argued that the validity of $\bigcirc(A/A)$ is not as paradoxical as we could think. This is, for example, the point of view of Prakken and Sergot [31]. Following one of Hansson's suggestions [15], they claim that the appearance of a paradox is considerably diminished if we assume that the antecedent of any dyadic obligation describes some settled or already fixed situation. Whilst acknowledging the importance of the notion of settleness in the literature [15, 12, 19, 33, 6], we prefer to leave this aside for the purposes of the present analysis.

2.1. Taking Futurity into Account

Makinson proceeds in two steps. He begins by formulating a concept of undefeasible conditional obligation, serving as a first approximation. He then extends the construction to cover defeasible obligation, stating the following:

> "$\bigcirc(B/A)$ is true in a world α iff in the most normal among the worlds β future to α in which A is true, B is true in all the best among the worlds γ that are in turn accessible from β".[1] [22, p. 373]

B true in all the best among the worlds γ that are in turn accessible from β, that is to say:

> "$\bigcirc B$ true in β" [22, p. 372].

The right hand side of the evaluation rule for $\bigcirc(/)$ mentions three elements. The first one is the normality relation. We will write $\alpha \prec \beta$ for "α is more normal than β". No particular

[1] The formulae of the quotation have been adapted to the notational convention used here.

constraints are placed on this relation [22, p. 345]. The second element is the temporal relation "– is future to – ". It is supposed to have "a logic bounded between S4 and S4.3" [22, p. 373]. In the first case, the relation is reflexive and transitive. In the second case, it is in addition future-connected: If $\alpha R\beta$ (to be read as "β is later than α") and $\alpha R\gamma$, then either $\beta R\gamma$ or $\gamma R\beta$. The third and final element introduced is the unconditional (or monadic) obligation. It is defined in terms of the deontic preference relation together with an auxiliary accessibility relation, as it is customary in dyadic deontic logic. Intuitively, this auxiliary relation is read as one of agency, linking two given worlds just when "it is possible to pass from [one] world [ ] to [another] by dint of human exertion" [22, pp. 358-359]. As defined above by Makinson, $\bigcirc B$ turns out to be equivalent to $\bigcirc(B/\top)$, where $\bigcirc(/)$ refers to Hansson's concept of conditional obligation. An obvious alternative would consist in interpreting the unconditional obligation operator by means of a deontic alternativeness relation, as in standard monadic deontic logic. In fact, as far as the validity of $\bigcirc(A/A)$ is concerned, the choice between the two interpretations will make no difference. We will, thus, leave the monadic operator defined in either way. Let $\|A\|$ denote the set of A-worlds, Fut_α be a notational shorthand for the set of worlds that are future to α and $\min(X, \prec)$ denote the set of $\prec$-minima of X. Makinson's truth-clause can be written as:

Definition 2.1. $\alpha \models \bigcirc(B/A)$ iff $\min(\|A\| \cap Fut_\alpha, \prec) \subseteq \| \bigcirc B\|$.

Intuitively, the procedure to determine the truth-value of $\bigcirc(B/A)$ at α goes as follows. First, intersect the set of worlds that are future to α with the set of those in which A is true. Next, list the elements from that intersection that are the most normal, and check whether they all contain the unconditional obligation $\bigcirc B$. If the response is yes, then $\bigcirc(B/A)$ is true in the actual world α. If the response is no, then it is false.

2.2. The Relation with Hansson

If this reformulation is correct, then Makinson shifts Hansson's initial perspective in two ways. First, unlike the latter, the former makes conditional obligation explicitly future-orientated: The obligation of B together with the condition A lie in the future relative to the world in which the conditional obligation is in force. Unlike van der Torre [36, p. 25], we do not think that Makinson's construction necessitates (to put it in van der Torre's terms) an "antecedent-before-consequent" interpretation of conditional norms. For the construction allows for every future antecedent-world to be contemporaneous with each of its best alternatives. It allows this, at least as a formal possibility.

There is another shift made by Makinson. Definition 2.1 analyzes the conditional (or dyadic) deontic operator in terms of the unconditional (or monadic) one, together with what Makinson refers to as a "normal future conditional" [22, p. 373], that we will write $\rightharpoondown$. This Lewis-type conditional connective is the "ontic" (or factual) counterpart of the conditional obligation operator, as defined above. To determine the truth value of $A \rightharpoondown B$, what you need to look at is not all the A-worlds that are the most normal, but only those that are future to the actual world. This makes $\bigcirc(B/A)$ equivalent to $A \rightharpoondown \bigcirc B$, as he observes [22, p. 373]. Clearly, it is a departure from Hansson's initial account, in which the dyadic operator is taken as "primitive" [15, p. 133] and as not being definable in terms of other connectives.

2.3. Futurity and Identity

We are now in a position to answer the question of whether the introduction of the futurity dimension is necessary to eliminate the identity principle. Let us imagine the following variations on definition 2.1, and compare what happens:

Definition 2.2. $\alpha \models \bigcirc(B/A)$ iff $\min(\|A\| \cap Fut_\alpha, \prec) \subseteq \|B\|$.

Definition 2.3. $\alpha \models \bigcirc(B/A)$ iff $\min(\|A\|, \prec) \subseteq \| \bigcirc B\|$.

Definition 2.2 is obtained from definition 2.1, by dropping the monadic obligation operator – so that the idea of obligation and conditionality are no longer separated from each other. Definition 2.3 is obtained from definition 2.1, by deleting the temporal accessibility relation. Note that, like in the theory of Chellas [8, 9] and Mott [27], such a definition makes $\bigcirc(B/A)$ equivalent to $A \Rightarrow \bigcirc B$, where $\Rightarrow$ is for the Lewis-type conditional connective, as it is usually defined. Note also that dropping the monadic obligation as well as the temporal relation yields Hansson's truth-clause:

Definition 2.4. $\alpha \models \bigcirc(B/A)$ iff $\min(\|A\|, \prec) \subseteq \|B\|$.

We recapitulate the result of this in a tabular form:

	$\bigcirc(/)$ primitive	$\bigcirc(/)$ future-orientated
Definition 2.4	yes	no
Definition 2.2	yes	yes
Definition 2.3	no	no
Definition 2.1	no	yes

The procedure to follow now is quite simple. To find out what the rejection of $\bigcirc(A/A)$ must be attributed to, it suffices to test the validity of the formula for each combination. We give the test results before going any further:

	$\bigcirc(/)$ primitive	$\bigcirc(/)$ future-orientated	$\bigcirc(A/A)$
Definition 2.4	yes	no	valid
Definition 2.2	yes	yes	valid
Definition 2.3	no	no	not valid
Definition 2.1	no	yes	not valid

The first two lines say that, whether futurity is taken into account or not, the use of monadic obligation is a necessary condition for the elimination of $\bigcirc(A/A)$. And the last two say that it is also a sufficient condition. Verification of the former is immediate. Definition 2.4 (Hansson's truth-clause) makes $\bigcirc(A/A)$ valid because, trivially,

$$\min(\|A\|, \prec) \subseteq \|A\|.$$

Exactly the same goes for definition 2.2:

$$\min(\|A\| \cap Fut_\alpha, \prec) \;\subseteq\; \|A\| \cap Fut_\alpha \;\subseteq\; \|A\|.$$

For the two other truth-clauses, the situation is different. And it is basically due to the fact that both represent obligation and conditionality by means of two distinct idioms. Consider first definition 2.3. From

$$\min(\|A\|, \prec) \subseteq \|A\|,$$

it does not follow that

$$\min(\|A\|, \prec)| \subseteq \| \bigcirc A\|,$$

because one may not have

$$\|A\| \subseteq \| \bigcirc A\|,$$

the failure of which is in accordance with our intuitions. It means that the model does not endorse as valid the sentence $A \supset \bigcirc A$, where $\supset$ is material implication. A similar line of reasoning applies to definition 2.1. There is no way to conclude

$$\min(\|A\| \cap Fut_\alpha, \prec) \subseteq \| \bigcirc A\|$$

from

$$\min(\|A\| \cap Fut_\alpha, \prec) \subseteq \|A\|.$$

2.4. Identity and Cumulativity: A Dilemma

The above way of avoiding the identity principle was first proposed by Chellas [8]. In the following, we focus on one of its drawbacks, and argue that it does not disappear with the introduction of the futurity dimension. This will lead us to a general consideration about the two approaches usually adopted for the analysis of ought-sentences evoked in the previous subsections. In the first account, the conditional obligation operator is taken as a primitive dyadic construction, with which a comparative goodness relation is associated. In the second, the conditional obligation operator is defined in terms of monadic obligation together with a Lewis-type conditional connective.

We usually expect a semantics to validate the following two principles, as corresponding to the weakest known reasonable alternatives for plain transitivity and full monotony:

$((A \Rightarrow B) \wedge ((A \wedge B) \Rightarrow C)) \supset (A \Rightarrow C)$	(Cut or cumulative transitivity)
$((A \Rightarrow B) \wedge (A \Rightarrow C)) \supset ((A \wedge B) \Rightarrow C)$	(Cautious or cumulative monotony)

Their deontic counterparts are respectively:

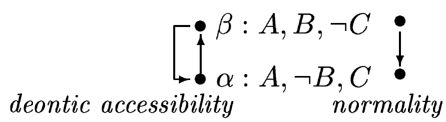
$(\bigcirc(B/A) \wedge \bigcirc(C/A \wedge B)) \supset \bigcirc(C/A)$	(Deontic cut)
$(\bigcirc(B/A) \wedge \bigcirc(C/A)) \supset \bigcirc(C/A \wedge B)$	(Deontic cautious monotony)

Intuitively, deontic cautious monotony says that already fulfilled obligations will not disturb further obligations. This principle has never been discussed in the literature. Deontic cut (also called "deontic detachment") tells us when we can chain obligations. This inference pattern has been discussed in the literature, only in the context of the contrary-to-duty scenarios [19, 6, 31, 37, 5]. We shall not attempt to enter into this discussion, which in general focusses on deontic detachment as opposed to other inference patterns, in particular factual detachment (or deontic modus ponens). In line with e.g. Nute and Yu [28, 29], we feel that the cut rule has strong intuitive support. We also believe that this line of thought is likewise applicable to cautious monotony. Now, it is not difficult to see that, if the dyadic obligation operator is taken as primitive, the deontic versions of cut and cautious monotony both hold. For the former to be validated, no particular constraint must be placed on the preference relation. By contrast, as studies [16] and [21] have shown, cautious monotony needs smoothness – an assumption that essentially forbids infinite sequences of ever more normal worlds. All we thus have to do is imposing a similar constraint on the comparative goodness relation. But suppose that the dyadic obligation operator is defined in terms of the monadic one together with the Lewis-type conditional connective. As can be easily verified, cut and cautious monotony both fail, whether futurity is introduced or not. To make this clear, we shall assume that the monadic operator is analyzed by means of the usual deontic alternativeness relation. As far as we can see, one would get the same result if it was defined in terms of comparative goodness together with agency.

Proposition 2.1. *If the truth conditions for $\bigcirc(/)$ are given as in definition 2.3 or as in definition 2.1, then the following two schemata can be falsified:*

$(\bigcirc(B/A) \wedge \bigcirc(C/A \wedge B)) \supset \bigcirc(C/A)$	*(Cut)*
$(\bigcirc(B/A) \wedge \bigcirc(C/A)) \supset \bigcirc(C/A \wedge B)$	*(Cautious monotony)*

Proof:
We begin with cut in a timeless framework. Consider a model in which $W = \{\alpha, \beta\}$, the sets of deontic alternatives to α and β are respectively $\{\beta\}$ and $\{\alpha\}$, $\prec$ is $\{(\alpha, \beta)\}$, $\|A\| = \{\alpha, \beta\}$, $\|B\| = \{\beta\}$ and $\|C\| = \{\alpha\}$.



From $\min(\|A\|, \prec) = \{\alpha\}$ and $\| \bigcirc B\| = \{\alpha\}$, it follows that $\alpha \models \bigcirc(B/A)$. From $\| \bigcirc C\| = \{\beta\}$ and $\min(\|A \wedge B\|, \prec) = \{\beta\}$, we obtain $\alpha \models \bigcirc(C/A \wedge B)$. Clearly, $\alpha \not\models \bigcirc(C/A)$, since $\alpha \not\models \bigcirc C$. To show that cut also fails for future-orientated obligation, it suffices to take the counterexample just given, and to put $Fut_\alpha = \{\alpha, \beta\}$ and $Fut_\beta = \{\beta\}$, so that $\min(\|A\| \cap Fut_\alpha, \prec) = \min(\|A\|, \prec)$ and $\min(\|A \wedge B\| \cap Fut_\beta, \prec) = \min(\|A \wedge B\|, \prec)$. This takes us back to the above refutation. We can analogously verify the failure of deontic cautious monotony, by simply changing the truth-value of C at α and β. The trouble here is that it is difficult to find an interpretation of the propositions which makes it intuitively clear why cautious monotony and cut fail. □

These brieve considerations by no means exhaust the question of how each approach compares with the other. A more systematic study of the properties satisfied by the operator that they generate remains to be done. We do not need here to pursue this investigation. Suffice it to notice that, as far as the identity principle and the cumulativity condition are concerned, we are apparently left in a quandary. For, according as to whether the dyadic obligation operator is taken as primitive or not, the two principles both hold or both fail. This consideration raises several interesting issues, among which are the following two:

- Is there any reason why we cannot develop an account satisfying the cumulativity condition but not the identity principle?

- What happens if, in Chellas-type definition of $\bigcirc(/)$, the operator $\Rightarrow$ is replaced by some other suitably choosen notion of conditionality?

We shall not pursue these issues here, but confine ourselves to indicating how the presence of cut and the absence of identity may interact. As a limiting case, cut gives $\bigcirc(\neg A/A) \supset (\bigcirc(B/A\wedge\neg A) \supset \bigcirc(B/A))$. In both traditional accounts of obligation, the formula $\bigcirc(B/A\wedge\neg A)$ is valid, since $\min(\|A \wedge \neg A\|, \prec) = \emptyset$, so that we finally get $\bigcirc(\neg A/A) \supset \bigcirc(B/A)$ – a law that is apparently harmless only within a framework in which the dyadic obligation operator is taken as primitive, since we can never have $\bigcirc(\neg A/A)$. This suggests a partial answer to the first issue we raised. One obvious possibility here is to block the validity of $\bigcirc(\neg A/A) \supset \bigcirc(B/A)$ by blocking that of $\bigcirc(B/A \wedge \neg A)$. But, to accomplish this, we need to go beyond the equipment usually envisaged for $\Rightarrow$ on the preference-based approach. For instance, one might pair the logic of $\Rightarrow$ with the Routley-Meyer semantics for relevant logic, as suggested in [11] and further elaborated in [24]. In the next section we shall explore an alternative solution, which consists in giving a new form to the cut rule. As a result of this modification, the obligation of destroying the state of affairs described by A, $\bigcirc(\neg A/A)$, will not imply that of replacing A by an arbitrary B, $\bigcirc(B/A)$.

3. Cumulativity without Identity

In the previous section, we have investigated the relationship between the futurity dimension and the identity principle, and seen that the introduction of the former is not indispensable in the elimination of the latter. This investigation has led us to a more general consideration about traditional theories of obligation, noticing that they usually leave us in a dilemma. According as to whether the dyadic obligation operator is taken as primitive or not, the identity principle and the cumulativity condition both hold or both fail. In the present section, we explore a way of getting out of such a dilemma.

So far, we have explored the possibility of adding the futurity dimension. We have noticed that, as far as the above dilemma is concerned, such an addition makes no difference. In the following, we go one step further in our study of the interaction of conditional obligation with time. We investigate what happens if, given a conditional norm of the form $\bigcirc(B/A)$, one also associates the earlier-to-later direction with the relation between A and B.

This section falls in two parts. In subsection 3.1, we propose a new account of defeasible conditional obligations, based on von Wright's logic of "and next" [40, 3]. In subsection 3.2, we turn to consider how this new account deals with the issue at stake here. The "and next" operator has been extensively studied by so-called dynamic deontic logic [26, 20, 10, 25]. In the sequel, we do not presume any familiarity with these studies.

3.1. Incorporating Temporal Precedence

3.1.1. Obligation and Sequention Combined

The basic idea is as follows. Defeasible conditional obligations may in general be defined as $A \wedge B > A \wedge \neg B$, where $>$ stands for some operator coming from the logic of preference. This suggests a simple way of incorporating temporal precedence. It consists in defining the dyadic obligation operator, not as a preference of $A \wedge B$ over $A \wedge \neg B$, but as a preference of $A;B$ over $A;\neg B$, with ; read as "and next":

$$\bigcirc(B/A) =_{def} A;B > A;\neg B.$$

There is no difficulty in reformulating the semantics in this fashion. Let $M = (Tree, <)$ be a temporal branching frame, where $Tree$ is a non-empty set of moments and $<$ is a tree-like ordering of these moments. Suppose the truth definition follows the pattern:

$$M, m/h \models A: \text{ "in model } M\text{, formula } A \text{ is true at moment } m \text{ in history } h\text{".}$$

Taking time to be discrete, we define:

$$M, m/h \models A;B \quad \text{iff} \quad M, m/h \models A \text{ and } M, m+1/h \models B.$$

It could justifiably be argued that the use of integers as time moments restricts the applicability of our framework to common-sense reasoning. For simplicity's sake, we shall leave this issue aside.

Turning now to the deontic component, we enrich the structure with a preference relation $\preceq$ on the set $\mathbf{H}$ of all histories. It is assumed that $\preceq$ is a pre-order (i.e. a reflexive and transitive relation). In addition it is also assumed that $\preceq$ is virtually connected in the sense that whenever $h \preceq h'$ then either $h \preceq h''$ or $h'' \preceq h'$. Notice that, in the presence of reflexivity, virtual connectivity implies so-called connectedness (either $h \preceq h'$ or $h' \preceq h$). Intuitively, this is a strong assumption. But, as we shall see, we need to place this constraint on the preference relation, so as to preserve the condition of cumulativity, at least in an adapted form.

We are now in a position to formulate the truth-clause for the dyadic obligation operator. Let H_m abbreviate the set of histories passing through m. We set:

Definition 3.1. $M, m/h \models \bigcirc(B/A)$ iff there exists a history $h' \in H_m$ such that (1) $M, m/h' \models A;B$ and (2) $h'' \not\preceq h'$ for any history $h'' \in H_m$ such that $M, m/h'' \models A;\neg B$.

This truth-clause says that, for $\bigcirc(B/A)$ to be true at an index m/h, there must be a history h' making $A;B$ true at m, such that for no history h'' making $A;\neg B$ true at m it is the case that $h'' \preceq h'$. This evaluation rule is typically preference-based, in the sense that we can define the dyadic obligation operator as

$$(A;B) > (A;\neg B),$$

for a suitably chosen preference operator $>$. To accomplish this, we just need to extend the ordering on histories to one on sets of histories using the following recipe: A set of histories is as good as a second one, if for each history in the second set there is a history in the first set which is as good as this one. Formally: $H \preceq^s H'$ (with the superscript "s" for "set") iff $(\forall h' \in H')(\exists h \in H)(h \preceq h')$. With $\preceq^s$ so defined, the evaluation rule to adopt for $>$ is, evidently, as follows:

Definition 3.2. $m/h \models A > B$ iff $\|B\|_m \not\preceq^s \|A\|_m$, with $\|A\|_m = \{h \in H_m : M, m/h \models A\}$.

In the sequel we often write $(A;B) > (A;\neg B)$ instead of $\bigcirc(B/A)$.

3.1.2. Minimality

Before going any further, there is an important remark to make. Although the notion of "most preferred" histories does not appear in the new truth conditions we propose, it can be shown that our approach has intimate connections with such a notion. This connection is made precise by proposition 3.1:

Lemma 3.1. *If H_m is finite, then $\preceq$ is "locally" stoppered in the following sense: for all formulas A and histories $h \in H_m$, if $h \in \|A\|_m$ then there is some $h' \in \|A\|_m$ such that $h' \preceq h$, and for all $h'' \in \|A\|_m$, $h'' \preceq h'$ implies $h' \preceq h''$.*

Proof:
This is easily proved by *reductio.* The verification is left to the reader. □

Proposition 3.1. *Let H_m be finite. Let A be locally consistent, i.e. $\|A\|_m \neq \emptyset$. The following two sentences are equivalent:*

(1) $(\exists h \in H_m)\{(M, m/h \models A; B) \ \&\ (\forall h' \in H_m)(M, m/h' \models A; \neg B \rightarrow h' \not\preceq h)\}$
(2) $\min(\|A\|_m, \preceq) \subseteq \|NB\|_m$,
where - $\min(\|A\|_m, \preceq) = \{h \in \|A\|_m : \forall h' \in \|A\|_m$ *if* $h' \preceq h$ *then* $h \preceq h'\}$
- *N stands for some unary next-time operator, defined in the obvious way*

Proof:
We first show that (1) implies (2). Let $h_1 \in \min(\|A\|_m, \preceq)$. Clearly, $h_1 \in \|A\|_m$, i.e. $M, m/h_1 \models A$. Let $h_1 \notin \|NB\|_m$, i.e. $M, m/h_1 \models A; \neg B$. By (1), $(\exists h)(M, m/h \models A \ \&\ h_1 \not\preceq h)$. By $h_1 \in \min(\|A\|_m, \preceq)$, $h \not\preceq h_1$. This contradicts the connectedness assumption. Thus, $h_1 \in \|NB\|_m$, as required.

Conversely, suppose $h_1 \in \|A\|_m$ – A locally consistent. Let H_m be finite. There is then some $h_2 \in \|A\|_m$ such that $h_2 \preceq h_1$, and for all $h \in \|A\|_m$, $h \preceq h_2$ implies $h_2 \preceq h$ – local stopperedness. So, $h_2 \in \min(\|A\|_m, \preceq)$ and, by (2), $M, m/h_2 \models A; B$. Consider any h_3. Suppose that $M, m/h_3 \models A$ and $h_3 \preceq h_2$. From this together with $h_2 \in \min(\|A\|_m, \preceq)$ and the transitivity of $\preceq$, it follows that $h_3 \in \min(\|A\|_m, \preceq)$. By (2), $M, m+1/h_3 \models B$. We thus have

$$(M, m/h_3 \models A \ \&\ h_3 \preceq h_2) \rightarrow M, m+1/h_3 \models B,$$

or, equivalently,

$$M, m/h_3 \models A; \neg B \rightarrow h_3 \not\preceq h_2.$$

This completes the proof. □

3.1.3. Related Work

The idea of incorporating temporal precedence into Hansson's initial account has been introduced and informally discussed by Spohn [32] and Alchourrón [2], as a way of eliminating the identity principle. In this respect, our contribution has been to provide a formalisation of their suggestion. Readers familiar with so-called conditional logics will have probably noticed that our two truth-clauses parallel that used by Lewis [17, 18] in a timeless context.

As observed by Makinson [23], real-life norms might exhibit many different "antecedent-before-consequent" (ABC) structures. In this paper, we focus on one of the simplest he mentions, at least as a starting point. Being shorthand for $A\,;B > A\,;\neg B$, the formula $\bigcirc(B/A)$ can be rendered as "at the present moment, if A holds at that point then at the next moment B is required to hold". One might extend the construction to cover several other ABC structures, by using so-called Priorean modalities and the like [4]. A systematic review of the different kinds of ABC structures that could be captured in that way remains to be done.

3.2. Identity and Cumulativity Revisited

It now remains to be seen whether these adjustments help us to solve the question we have been dealing with.

An easy example can show that $\bigcirc(A/A)$ is no longer valid. For a counterexample, we only need to choose a history h' in which A (e.g. "the window is open") is true at m but false at $m+1$, and such that for no histories h'' in which A is true at m and $m+1$, it is the case that $h'' \preceq h'$. In this case, $M, m/h \models (A\,;\neg A) > (A\,;A)$ or, equivalently, $M, m/h \models \bigcirc(\neg A/A)$. We now turn our attention to cautious monotony and cut, beginning with the former.

Proposition 3.2. *If the truth-conditions for $\bigcirc(/)$ are given as in definition 3.1, then the following schema is valid:*

$$(\bigcirc(B/A) \wedge \bigcirc(C/A)) \supset \bigcirc(C/A\,;B).$$

This schema resembles cautious monotony, but its formulation uses ; instead of $\wedge$. Literally speaking, the conclusion $\bigcirc(C/A\,;B)$ says that:

$$(A\,;B)\,;C > (A\,;B)\,;\neg C.$$

This expression must be read with care. To see its significance, it is helpful to look at an equivalent formulation. It is:

$$A;(B \wedge C) > A;(B \wedge \neg C).$$

This second expression says that, given the truth of A at the present moment, doing B together with C at the next moment is better than doing B but not C.

Proof:
Suppose that $M, m/h \models A\,;B > A\,;\neg B$ and that $M, m/h \models A\,;C > A\,;\neg C$, I.e.

- there exists a history $h_1 \in H_m$ such that (1a) $M, m/h_1 \models A\,;B$ and (1b) $h \not\preceq h_1$ for any history $h \in H_m$ such that $M, m/h \models A\,;\neg B$
- there exists a history $h_2 \in H_m$ such that (2a) $M, m/h_2 \models A\,;C$ and (2b) $h' \not\preceq h_2$ for any history $h' \in H_m$ such that $M, m/h' \models A\,;\neg C$.

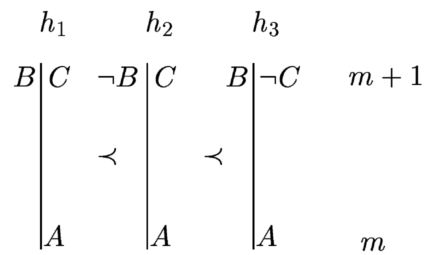
We argue as follows. If $M, m+1/h_2 \models B$, then h_2 satisfies the conditions for $A\,;(B \wedge C) > A\,;(B \wedge \neg C)$ – or, equivalently, $(A\,;B)\,;C > (A\,;B)\,;\neg C$. If $M, m+1/h_2 \not\models B$, then h_1 satisfies the conditions for $A\,;(B \wedge C) > A\,;(B \wedge \neg C)$:

Case 1. $M, m+1/h_2 \models B$. In this case, $M, m/h_2 \models A; (B \wedge C)$. By (2b), we have $h' \not\preceq h_2$ for any history $h' \in H_m$ such that $M, m/h' \models A; \neg C$. From this, it follows that $h'' \not\preceq h_2$ for any history $h'' \in H_m$ such that $M, m/h'' \models A; (B \wedge \neg C)$, as desired.

Case 2. $M, m+1/h_2 \not\models B$. In this case, $M, m/h_2 \models A; \neg B$. By (1b), $h_2 \not\preceq h_1$. By (2b), if $M, m+1/h_1 \not\models C$ then we would have $h_1 \not\preceq h_2$, contradicting $h_2 \not\preceq h_1$ – as a limiting case, virtual connectivity yields connectedness. Therefore, $M, m+1/h_1 \models C$ and, as a result, $M, m/h_1 \models A; (B \wedge C)$. Consider any $h'' \in H_m$ such that $M, m/h'' \models A; (B \wedge \neg C)$. Clearly, $M, m/h'' \models A; \neg C$. By (2b), $h'' \not\preceq h_2$. From this together with $h_2 \not\preceq h_1$, it follows that $h'' \not\preceq h_1$ – virtual connectivity of $\preceq$.

□

To show that the assumption of virtual connectivity cannot be dropped, it is more transparent to use a strict (irreflexive) preference relation $\prec$, defined as the converse complement of $\preceq$: $h \prec h'$ iff $h' \not\preceq h$. Virtual connectivity of $\preceq$ coincides with (is equivalent to) a property of $\prec$ which is much easier to visualize. It is the property of transitivity. Now, consider:



From $h_1 \prec h_2$ ($h_2 \not\preceq h_1$), we infer $M, m/h_i \models A; B > A; \neg B$ for $i \in \{1, 2, 3\}$. From $h_2 \prec h_3$ ($h_3 \not\preceq h_2$), we infer $M, m/h_i \models A; C > A; \neg C$. If $h_1 \not\prec h_3$ ($h_3 \preceq h_1$), we obtain $M, m/h_i \not\models A; (B \wedge C) > A; (B \wedge \neg C)$. If $h_1 \prec h_3$ ($h_3 \not\preceq h_1$), we get $M, m/h_i \models A; (B \wedge C) > A; (B \wedge \neg C)$.

Proposition 3.3. *If the truth-conditions for $\bigcirc(/)$ are given as in definition 3.1, then the following adapted version of cumulative transitivity is validated:*

$$(\bigcirc(B/A) \wedge \bigcirc(C/A; B)) \supset \bigcirc(C/A). \tag{1}$$

Proof:

Suppose that $M, m/h \models A; B > A; \neg B$, $M, m/h \models A; (B \wedge C) > A; (B \wedge \neg C)$ and $M, m/h \not\models A; C > A; \neg C$. The assumption $M, m/h \models A; (B \wedge C) > A; (B \wedge \neg C)$ means that there exists a history $h_1 \in H_m$ such that $M, m/h_1 \models A; (B \wedge C)$ and

$$(\forall h \in H_m)(M, m/h \models A; (B \wedge \neg C) \rightarrow h \not\preceq h_1) \tag{2}$$

Clearly, $M, m/h_1 \models A; C$. From this together with $M, m/h \not\models A; C > A; \neg C$, it follows that there exists a history $h_2 \in H_m$ such that $M, m/h_2 \models A; \neg C$ and

$$h_2 \preceq h_1 \tag{3}$$

We need to consider two cases:

Case 1. $M, m+1/h_2 \models B$. In this case, $M, m/h_2 \models A; (B \wedge \neg C)$. By (2), we conclude that $h_2 \not\preceq h_1$ – contradicting $h_2 \preceq h_1$.

Case 2. $M, m+1/h_2 \not\models B$. In this case, $M, m/h_2 \models A; \neg B$. The assumption $M, m/h \models A; B > A; \neg B$ means that there exists a history $h_3 \in H_m$ such that $M, m/h_3 \models A; B$ and

$$(\forall h \in H_m)(M, m/h \models A; \neg B \rightarrow h \not\preceq h_3). \tag{4}$$

Thus, $h_2 \not\preceq h_3$. Once again, we need to consider two cases:

Case 2.1. $M, m+1/h_3 \not\models C$. Clearly, $M, m/h_3 \models A; (B \wedge \neg C)$. By (2), we conclude that $h_3 \not\preceq h_1$. From this together with $h_2 \not\preceq h_3$, it follows (by virtual connectivity) that $h_2 \not\preceq h_1$, contradicting (3).

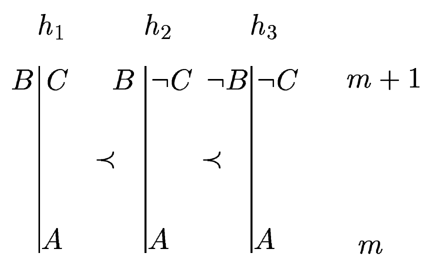
Case 2.2. $M, m+1/h_3 \models C$. In this case, $M, m/h_3 \models A; C$. From this together with $M, m/h \not\models A; C > A; \neg C$, it follows that there is a history $h_4 \in H_m$ such that $M, m/h_4 \models A; \neg C$ and $h_4 \preceq h_3$. We need again to consider two cases:

Case 2.2.1. $M, m+1/h_4 \not\models B$. In this case, $M, m/h_4 \models A; \neg B$. By (4), $h_4 \not\preceq h_3$, a contradiction.

Case 2.2.2. $M, m+1/h_4 \models B$. In this case, $M, m/h_4 \models A; (B \wedge \neg C)$. By (2), $h_4 \not\preceq h_1$. But from $h_4 \preceq h_3$ together with $h_2 \not\preceq h_3$, it follows, by virtual connectivity, that $h_4 \preceq h_2$ which, by transitivity, implies $h_4 \preceq h_1$. Contradiction.

□

We have seen that cautious monotony may fail unless $\preceq$ is required to be virtually connected. To show that exactly the same goes for cut, we simply need to change some truth-values in the countermodel given to cautious monotony:



From $h_2 \prec h_3$ (i.e. $h_3 \not\preceq h_2$), it follows that $M, m/h_i \models A; B > A; \neg B$ for $i = 1, 2, 3$. From $h_1 \prec h_2$ ($h_2 \not\preceq h_1$), we get $M, m/h_i \models A; (B \wedge C) > A; (B \wedge \neg C)$. If $h_1 \not\prec h_3$ ($h_3 \preceq h_1$), then $M, m/h_i \not\models A; C > A; \neg C$. But if $h_1 \prec h_3$ ($h_3 \not\preceq h_1$), then $M, m/h_i \models A; C > A; \neg C$.

4. Conclusion

The main focus of this article was that the usual accounts of conditional obligation leave us with a quandary: We have no choice but to accept both the identity principle and the cumulativity condition, or none. In an attempt to get out of this quandary, we have proposed an analysis of defeasible conditional obligation in terms of the "and next" operator. In this account, an adapted form of cumulativity is preserved, and the identity principle fails.

However, it can justifiably be argued that the dilemma remains to be resolved. First, we have seen that, in order to preserve the cumulativity condition, it is necessary to require the preference relation to be virtually connected. Some might rightly ask if, by itself, this is not too strong an assumption. In fact, virtual connectivity may be dropped, if the truth-conditions for the dyadic obligation operator are given a form similar to that used in [14] for the ontic conditional operator:

Definition 4.1. $M, m/h \models \bigcirc(B/A)$ iff for all $h' \in H_m$ such that $M, m/h' \models A;\neg B$, there exists a history $h'' \in H_m$ such that (1) $M, m/h'' \models A;B$ (2) $h'' \prec h'$ and (3) for all h' such that $M, m/h' \models A;\neg B$, if $h' \preceq h''$ then $h'' \preceq h'$.

Here $\prec$ stands for the "strengthened" converse complement of $\preceq$ defined by putting $h \prec h'$ iff both $h \preceq h'$ and not $h' \preceq h$. Our sightly revised form of cautious monotony is now valid in any semantics that only assumes transitivity for the relation $\preceq$. For cut, the additional assumption that the set of histories is finite seems to be necessary. We omit the verifications, because such a variation on the way in which a defeasible conditional obligation is evaluated would be of no help to cope with another difficulty which our account faces. We clearly have:

$$\begin{aligned} (A;C);B &\equiv (A;C);(B \wedge C), \\ (A;C);\neg B &\equiv (A;C);\neg(B \wedge C). \end{aligned}$$

It immediatly follows that, if the truth-conditions for $\bigcirc(/)$ are given as in definition 3.1 or as in definition 4.1, the following adapted form of identity is validated:

$$\bigcirc(B/A;C) \supset \bigcirc(B \wedge C/A;C).$$

We shall pursue this final difficulty in future research.

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